CS 683: Advanced Algorithms	LP duality
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Consider the following scheduling problem. There are m distinct machines and n customers. Customers rent machine time to perform jobs. Each machine can serve only one customer at a time. Each customer i specifies a set S_i of machines that (s)he needs; all machines in S_i get the same amount (number of minutes) of work from i. Each machine accepts at most T mins of work. It's OK to permute jobs (customers) on a given machine. The revenue is $\$1/min$ from every customer, independently of how many machines the customer uses. The goal is to determine the amounts of time x_i to rent to customers to maximize the revenue.

This problem can be interpreted as a task-scheduling problem on a supercomputer, where each task requires a certain subset of devises (e.g. CPU, satellite link and a color printer). Alternatively, it could be resource-management problem in industry, where "customers" stand for factories, and "machines" stand for resources the factories use (e.g. steel, wood and coal).

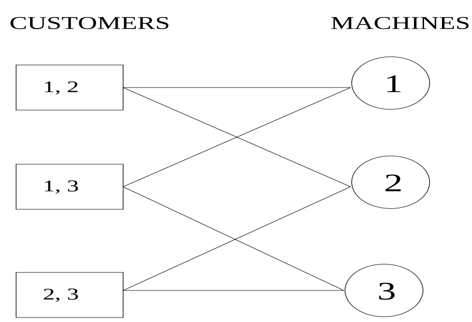

Figure 1: Three customers with needs $\{1,2\},\{1,3\},\{2,3\}$, resp.

Example 1 *Consider 3 machines and 3 customers with needs* $\{1,2\},\{1,3\},\{2,3\}$, *resp. (see Fig. 1). Allocating* $\frac{1}{2}T$ *mins to every customer saturates all resources, hence is feasible. Since every* $\$1$ *from anyone burns up* 2 *mins of resources, this allocation is optimal.*

Example 2 *Now consider 4 machines and 3 customers with needs* $\{1,2\},\{1,3\},\{1,4\}$, *resp. (see Fig. 2). Now machine* 1 *is the bottleneck: every* $\$1$ *of revenue burns up* 1 *min on machine* 1*. Thus, the revenue cannot exceed* T*. The revenue of* T *exactly is achieved by allocating* T *mins to any one customer.*

How can we prove in general that a revenue of P is not achievable? We need to assign to each resource j its *intrinsic value* $y_j \geq 0$ so that:

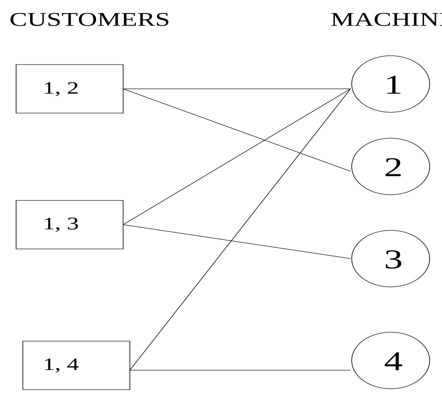

Figure 2: Three customers with needs $\{1,2\},\{1,3\},\{1,4\}$, resp.

1. Every dollar you make burns up at least 1 unit of value: for every customer i,

$$\sum_{j\in S_i} y_i \geq 1$$

.

2. The total intrinsic value is small:

$$\sum_{y=1}^{m} y_j T < P$$

It is intuitively clear that if the above conditions hold, one can't make revenue P. Indeed,

$$\sum_{i=1}^{n} x_i \;\leq\; \sum_{i=1}^{n} x_i \left(\sum_{j\in S_i} y_j\right) \;=\; \sum_{j=1}^{m} y_j \left(\sum_{i:j\in S_i} x_i\right) \;\leq\; \sum_{j=1}^{m} y_j T \;<\; P$$

In Ex. 1, $y_1 = y_2 = y_3 = 1/2$; in Ex. 2, $y_1 = 1$ and $y_2 = y_3 = y_4 = 0$.

Now, we prove the same result with matrices and vectors, which will allow us to observe the customer and machine problems as linear problems. So, we construct a mapping from customer $i \mapsto$ vector s_i given by

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$$

where the jth entry is 1 if $j \in S_i$, and 0 otherwise. The matrix A is then given by

$$\left[\begin{array}{c|c|c} s_1 & s_2 & \ldots \end{array}\right]$$

and we are looking for a choice vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

with $x_i \geq 0$, such that (using the coordinate-wise less than or equal operator)

$$A \cdot x \leq_{cw} \begin{bmatrix} T \\ T \\ \vdots \end{bmatrix} = \vec{T}$$

The goal is, of course, to find the $\max \sum_i x_i = \vec{1} \cdot x$. The two prior conditions for showing the impossibility of P, are now rephrased as $y \cdot A \geq_{cw} \vec{1}$ and $y \cdot \vec{T} < P$, where $y = \begin{bmatrix} y_1 & y_2 & \ldots \end{bmatrix}$.

Let the optimum revenue be denoted by P^*, achieved by the optimum choice vector x^*. Then the LP-duality theorem can be given as

Theorem 3 *If $P^* < P$, then $\exists y \geq_{cw} \vec{0}$, such that $y \cdot A \geq_{cw} \vec{1}$ and $y \cdot \vec{T} < P$.*
Or $P^* = \min y \cdot \vec{T}$, such that $y \cdot A \geq_{cw} \vec{1}$ and $y \geq_{cw} \vec{0}$.

Proof. (in next lecture) ■

This preliminary version of LP-duality asks: Given a matrix A and a vector b, when does $A \cdot x \leq b$ have no solution? Notice the reminiscence to $A \cdot x = b$, solved by Gaussian elimination.