Machine Learning Theory (CS 6783)

Lecture 13 : Bit Prediction and Multiclass Prediction

1 Bit Prediction

Claim 1. *There exists a randomized prediction strategy that ensures that*

$$\mathbb{E}\left[\mathrm{Reg}_n\right] \le \frac{1}{2n}\mathbb{E}_\epsilon\left[\sup_{f\in\mathcal{F}}\sum_{t=1}^{n} f_t\epsilon_t\right]$$

To prove the above claim we first prove this following lemma, a result by Thomas Cover.

Lemma 2 (T. Cover'65)**.** *Let $\phi : \{\pm1\}^n \mapsto \mathbb{R}$ be a function such that, for any i, and any $y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n$,*

$$|\phi(y_1,\ldots,y_{i-1},+1,y_{i+1},\ldots,y_n) - \phi(y_1,\ldots,y_{i-1},-1,y_{i+1},\ldots,y_n)| \le \frac{1}{n}\ \text{, (stability condition)}$$

then, there exists a randomized strategy such that for any sequence of bits,

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\hat{y}_t\sim q_t}\left[\mathbf{1}\{\hat{y}_t \neq y_t\}\right] \le \phi(y_1,\ldots,y_n)$$

if and only if,

$$\mathbb{E}_\epsilon\phi(\epsilon_1,\ldots,\epsilon_n) \ge \frac{1}{2}$$

and further, the strategy achieving this bound on expected error is given by:

$$q_t = \frac{1}{2} + \frac{n}{2}\ \mathbb{E}_{\epsilon_{t+1},\ldots,\epsilon_n}\left[\phi(y_1,\ldots,y_{t-1},-1,\epsilon_{t+1},\ldots,\epsilon_n) - \phi(y_1,\ldots,y_{t-1},+1,\epsilon_{t+1},\ldots,\epsilon_n)\right]$$

Proof of Lemma.
We start by proving that if there exists an algorithm that guarantees that

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\hat{y}_t\sim q_t}\left[\mathbf{1}\{\hat{y}_t \neq y_t\}\right] \le \phi(y_1,\ldots,y_n)$$

then, $\mathbb{E}_\epsilon\left[\phi(\epsilon_1,\ldots,\epsilon_n)\right] \ge 1/2$**.**

To see this, note that the regret bound implies that

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\hat{y}_t\sim q_t}\left[\mathbf{1}\{\hat{y}_t \neq y_t\}\right] - \phi(y_1,\ldots,y_n) \le 0$$

for any $y_1, \ldots, y_n$. Now simply let the adversary pick $y_t = \epsilon_t$ as a Rademacher random variable. Thus, taking expectation, this implies that,

$$0 \geq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}_{\hat{y}_t \sim q_t} \left[\mathbb{E}_{\epsilon_t} \mathbf{1}\{\hat{y}_t \neq \epsilon_t\} \right] - \mathbb{E}_{\epsilon} \phi(\epsilon_1, \ldots, \epsilon_n) = \frac{1}{2} - \mathbb{E}_{\epsilon} \phi(\epsilon_1, \ldots, \epsilon_n)$$

Next we prove that if $\mathbb{E}_{\epsilon}\phi(\epsilon_1, \ldots, \epsilon_n) \geq \frac{1}{2}$, then $\exists$ strategy s.t. $\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}_{\hat{y}_t \sim q_t}\left[\mathbf{1}\{\hat{y}_t \neq y_t\}\right] \leq \phi(y_1, \ldots, y_n)$.

The basic idea is to prove this statement starting from n and moving backwards. Say we have already played rounds up until round $n-1$ and have observed $y_1, \ldots, y_{n-1}$. Now let us consider the last round. On the last round we use,

$$q_n = \frac{1}{2} + \frac{n}{2}\ \phi(y_1, \ldots, y_{n-1}, -1) - \phi(y_1, \ldots, y_{n-1}, +1)$$

Now note that if $y_n = +1$ then $\mathbb{E}_{\hat{y}_n \sim q_n}\left[\ \mathbb{1}_{\{\hat{y}_n \neq y_n\}}\right] = \mathbb{E}_{\hat{y}_n \sim q_n}\left[\ \mathbb{1}_{\{\hat{y}_n = -1\}}\right] = 1 - q_n$ and if $y_n = -1$ then $\mathbb{E}_{\hat{y}_n \sim q_n}\left[\ \mathbb{1}_{\{\hat{y}_n \neq y_n\}}\right] = q_n$ and hence for the choice of q_n above, we can write

$$\mathbb{E}_{\hat{y}_n \sim q_n}\left[\ \mathbb{1}_{\{\hat{y}_n \neq y_n\}}\right] = \tfrac{1}{2n} - \tfrac{y_n}{2}\left(\phi(y_1, \ldots, y_{n-1}, -1) - \phi(y_1, \ldots, y_{n-1}, +1)\right)$$

Plugging in the above, note that for any y_n (possibly chosen adversarially looking at q_n), we have,

$$\begin{aligned}
&\tfrac{1}{n}\mathbb{E}_{\hat{y}_n \sim q_n}\left[\ \mathbb{1}_{\{\hat{y}_n \neq y_n\}}\right] - \phi(y_1, \ldots, y_n) && (1)\\
&\qquad = \frac{1}{2n} - \frac{y_n}{2}\left(\phi(y_1, \ldots, y_{n-1}, -1) - \phi(y_1, \ldots, y_{n-1}, +1)\right) - \phi(y_1, \ldots, y_n)\\
&\qquad = \frac{1}{2n} - \frac{1}{2}\left(\phi(y_1, \ldots, y_{n-1}, -1) + \phi(y_1, \ldots, y_{n-1}, +1)\right)\\
&\qquad = \frac{1}{2n} - \mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-1}, \epsilon_n) && (2)
\end{aligned}$$

Now recursively we continue just as above for $n-1$ to 0. Let us do the $n-1$th step and the rest follows. To this end, note that just as earlier, if $y_{n-1} = +1$ then $\mathbb{E}_{\hat{y}_{n-1} \sim q_{n-1}}\left[\ \mathbb{1}_{\{\hat{y}_{n-1} \neq y_{n-1}\}}\right] = \mathbb{E}_{\hat{y}_{n-1} \sim q_{n-1}}\left[\ \mathbb{1}_{\{\hat{y}_{n-1} = -1\}}\right] = 1 - q_{n-1}$ and if $y_{n-1} = -1$ then $\mathbb{E}_{\hat{y}_{n-1} \sim q_{n-1}}\left[\ \mathbb{1}_{\{\hat{y}_{n-1} \neq y_{n-1}\}}\right] = q_{n-1}$ and hence for the choice of $q_{n-1} = \frac{1}{2n} + \frac{n}{2}\ \mathbb{E}_{\epsilon_n}\left[\phi(y_1, \ldots, y_{n-2}, -1, \epsilon_n) - \phi(y_1, \ldots, y_{n-2}, +1, \epsilon_n)\right]$, we have

$$\tfrac{1}{n}\mathbb{E}_{\hat{y}_{n-1} \sim q_{n-1}}\left[\ \mathbb{1}_{\{\hat{y}_{n-1} \neq y_{n-1}\}}\right] = \tfrac{1}{2n} - \tfrac{y_{n-1}}{2}\left(\mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-2}, -1, \epsilon_n) - \mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-2}, +1, \epsilon_n)\right)$$

Thus we can conclude that,

$$\begin{aligned}
&\tfrac{1}{n}\mathbb{E}_{\hat{y}_{n-1} \sim q_{n-1}}\left[\ \mathbb{1}_{\{\hat{y}_{n-1} \neq y_{n-1}\}}\right] + \tfrac{1}{n}\mathbb{E}_{\hat{y}_n \sim q_n}\left[\ \mathbb{1}_{\{\hat{y}_n \neq y_n\}}\right] - \phi(y_1, \ldots, y_n)\\
&\quad = \frac{1}{2n} + \tfrac{1}{n}\mathbb{E}_{\hat{y}_{n-1} \sim q_{n-1}}\left[\ \mathbb{1}_{\{\hat{y}_{n-1} \neq y_{n-1}\}}\right] - \mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-1}, \epsilon_n) \qquad \text{(From Eq.2)}\\
&\quad = \frac{2}{2n} - \tfrac{y_{n-1}}{2}\left(\mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-2}, -1, \epsilon_n) - \mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-2}, +1, \epsilon_n)\right) - \mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-1}, \epsilon_n)\\
&\quad = \frac{2}{2n} - \frac{1}{2}\left(\mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-2}, +1, \epsilon_n) + \mathbb{E}_{\epsilon_n}\phi(y_1, \ldots, y_{n-2}, -1, \epsilon_n)\right)\\
&\quad = \frac{2}{2n} - \mathbb{E}_{\epsilon_{n-1}, \epsilon_n}\phi(y_1, \ldots, y_{n-2}, \epsilon_{n-1}, \epsilon_n)
\end{aligned}$$

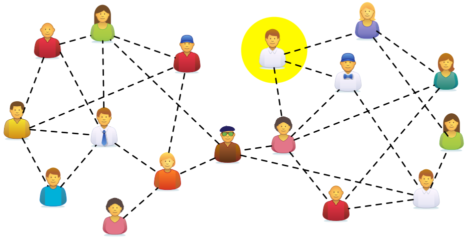
Proceeding in similar way we conclude that,

$$\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\hat{y}_t\sim q_t}\left[\mathbb{1}_{\{\hat{y}_t\neq y_t\}}\right]-\phi(y_1,\dots,y_n)\leq\frac{n}{2n}-\mathbb{E}_{\epsilon_1,\dots,\epsilon_n}\phi(\epsilon_1,\dots,\epsilon_n)=\frac{1}{2}-\mathbb{E}_{\epsilon_1,\dots,\epsilon_n}\phi(\epsilon_1,\dots,\epsilon_n)$$

Hence, if $\mathbb{E}_{\epsilon_1,\dots,\epsilon_n}\phi(\epsilon_1,\dots,\epsilon_n)\geq 1/2$ then we can conclude that, $\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}_{\hat{y}_t\sim q_t}\left[\mathbb{1}_{\{\hat{y}_t\neq y_t\}}\right]\leq\phi(y_1,\dots,y_n)$ as desired.

Hence we conclude the proof of this lemma. □

2 Application: Binary Node Classification



Let $G=(V,E)$ be a known undirected graph representing a social network. At each time step t, a user in the network opens her Facebook page, and the system needs to decide whether to classify the user as type "-1" or "$+1$", say, in order to decide on an advertisement to display. We assume here that the feedback on the "correct" type is revealed to the system after the prediction is made. Suppose we have a hunch that the type of the user ($+1$ or -1) is correlated with the community to which she belongs. For simplicity, suppose there are two communities, more densely connected within than across. To capture the idea of correlating communities and labels, we set ϕ to be small on labelings that assign homogenous values within each community. We make the following simplifying assumptions: (i) $|V|=n$, (ii) we only predict the label of each node once, and (iii) the order in which the nodes are presented is fixed (this assumption is easily removed). Smoothness of a labeling $f\in\{\pm1\}^n$ with respect to the graph may be computed via

$$\mathrm{Cut}(f)=\sum_{(u,v)\in E}\mathbb{1}_{\{f_u\neq f_v\}}=\frac{1}{4}\sum_{(u,v)\in E}(f_u-f_v)^2=f^\top Lf \tag{3}$$

where $L=D-A$, the diagonal matrix D contains degrees of the nodes, and A is the adjacency matrix and $f_v\in\{\pm1\}$ is the label in f that corresponds to vertex $v\in V$. This function in (3) counts the number of disagreements in labels at the endpoints of each edge. The value is also known as the size of the cut induced by f (the smallest possible being MinCut). As desired, the function in (3) gives a smaller value to the labelings that are homogenous within the communities.

Unfortunately, the function $\mathrm{Cut}(f)$ is not stable. Further, the cut size is $n-1$ for a star graph, where $n-1$ nodes, labeled as $+1$, are connected to the center node, labeled as -1. The large value of the cut does not capture the simplicity of this labeling, which is only one bit away from being a constant $+1$. Instead, we opt for the indirect definition:

$$F_\kappa = \Big\{ f \in \{\pm 1\}^n \ : \ f^\top L f \le \kappa \Big\} \tag{4}$$

for $\kappa \ge 0$, and then set

$$\phi(y_1, \dots, y_n) = \inf_{f \in \mathcal{F}_\kappa} \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}_{\{f_t \neq y_t\}} + \frac{1}{2n} \mathbb{E}_\epsilon \left[\sup_{f \in \mathcal{F}_\kappa} \sum_{t=1}^{n} f_t \epsilon_t \right] \tag{5}$$

Parameter κ should be larger than the value of MinCut, for otherwise the set F_κ is empty. This gives an interesting algorithm for the prediction problem .... What does this look like?

Well we want to use the strategy

$$\begin{aligned} q_t &= \frac{1}{2} + \frac{n}{2}\ \mathbb{E}_{\epsilon_{t+1},\dots,\epsilon_n} \left[\phi(y_1, \dots, y_{t-1}, -1, \epsilon_{t+1}, \dots, \epsilon_n) - \phi(y_1, \dots, y_{t-1}, +1, \epsilon_{t+1}, \dots, \epsilon_n) \right] \\ &= \frac{1}{2} + \frac{n}{2}\ \mathbb{E}_{\epsilon_{t+1},\dots,\epsilon_n} \left[\inf_{f \in \mathcal{F}_\kappa} \left\{ \frac{1}{n} \sum_{j=1}^{t-1} \mathbb{1}_{\{f_j \neq y_j\}} + \mathbb{1}_{\{f_t \neq -1\}} + \sum_{j=t+1}^{n} \mathbb{1}_{\{f_j \neq \epsilon_j\}} \right\} \right. \\ &\qquad \left. - \inf_{f \in \mathcal{F}_\kappa} \left\{ \frac{1}{n} \sum_{j=1}^{t-1} \mathbb{1}_{\{f_j \neq y_j\}} + \mathbb{1}_{\{f_t \neq +1\}} + \sum_{j=t+1}^{n} \mathbb{1}_{\{f_j \neq \epsilon_j\}} \right\} \right] \end{aligned}$$

It turns out that by concentration inequalities, it even suffices to take a single new sample of $\epsilon_{t+1}, \dots, \epsilon_n$ for round t to compute q_t above. In this case the underlying strategy is peculiar: At time t, to predict label for vertex v_t, we fill seen entries by labels, unseen entries by random ϵ_v's and solve two optimization problems. One with labels set as mentioned and with label of v_t set to -1 we solve for $\inf_{f \in \mathcal{F}_\kappa} \left\{ \frac{1}{n} \sum_{j=1}^{t-1} \mathbb{1}_{\{f_j \neq y_j\}} + \mathbb{1}_{\{f_t \neq -1\}} + \sum_{j=t+1}^{n} \mathbb{1}_{\{f_j \neq \epsilon_j\}} \right\}$. Now we do the optimization with only changing the label of v_t to a $+1$. We can then set q_t by equation above. Here once can view the random signs we draw as a kind of regularization or protection against worst case adversarial future.

Of course two natural questions follow. First, what if outcomes are not binary. We will see this in the following section. Second, what if we did not know the graph in advance or worse yet the graph evolves with time, or more generally what if we didnt have just bit prediction but rather prediction of bit given some input x_t like in the classification setting?

3 A Game of Betting

Consider a gambler who bets on the outcomes of games one every round. Specifically, on any round t, the gambler can choose an amount $|\widehat{y}_t|$ to bet on the outcome of game between two players or teams A and B. The gambler can choose to place this bet of $|\widehat{y}_t|$ on either team A to win or on team B. If the chosen team wins, the gambler gains an additional amount of $\widehat{y}_t$ and if the chosen team looses the gambler looses the bet amount of $\widehat{y}_t$. This game of betting can be formalized as the following linear game between the gambler and the house. Specifically, we can view the choice of the gambler at round t as a real number $\widehat{y}_t$. The magnitude $\widehat{y}_t$ denotes the bet amount and the sign of $\widehat{y}_t$ denotes whether the bet is placed on team A or team B. The corresponding outcome of the game is encoded by the variable $y_t \in \{\pm 1\}$ which indicates whether team A won or team B. At time t, $-\widehat{y}_t \cdot y_t$ denotes the loss of the gambler. That is if the gambler guessed the outcome right, that is if $\mathrm{sign}(\widehat{y}_t) = y_t$, then the loss is the negative value of $-|\widehat{y}_t|$ (or in other words the gambler gains) and if the outcome is guessed in correctly the gambler looses the amount of $|\widehat{y}_t|$.

> At time $t = 1, \ldots, n$, the forecaster chooses $\widehat{y}_t \in \mathbb{R}$ based on the history $y_1, \ldots, y_{t-1}$ and then observes the value $y_t \in \{\pm 1\}$.

Given some benchmark function $\phi : \{\pm 1\}^n \to \mathbb{R}_{\geq 0},$, the goal of the gambler is to ensure that the loss of the gambler is smaller than this benchmark. In other words, the gambler would like to ensure that,

$$\forall \boldsymbol{y}, \quad \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} -\widehat{y}_t y_t\right] \leq \phi(\boldsymbol{y}) \tag{6}$$

Lemma 3. *ϕ is achievable if and only if $\mathbb{E}\left[\phi(\boldsymbol{\varepsilon})\right] \geq 0$. Further, in this case, the strategy for the gambler is given by: $\widehat{y}_t = n \cdot \mathbb{E}[\phi(y_{1:t-1}, -1, \varepsilon_{t+1:n}) - \phi(y_{1:t-1}, +1, \varepsilon_{t+1:n})]$.*

Remark: stability is not required.

Example 3.1. *We have a gambler who likes to bet on games played between m teams. Assume that the information about which pairs of teams play each other for the n matches is announced in advance. Specifically, say we know that on round t, teams i_t and j_t play each other. Let us further denote by n_i the number of games played by player i. This game of betting can be formalized in the linear betting games framework above. As specific benchmark a gambler might consider is the one where each of the m team is given a score represented by an m dimensional vector $\boldsymbol{w}$. Further, when team i plays team j, a bet of amount of $|w[i] - w[j]|$ on the team with the larger score is placed. Further, assume that the largest bet amount is restricted to B. The goal of the gambler is to do as well as the best scoring of the teams selected in hindsight. This example, can be represented by the benchmark $\phi\{\pm 1\}^n \mapsto \mathbb{R}$ as follows:*

$$\phi(y_1, \ldots, y_n) = \inf_{\boldsymbol{w} \in \mathbb{R}^m : \max_{i,j} \boldsymbol{w}[i] - \boldsymbol{w}[j] \leq B} \frac{1}{n}\sum_{t=1}^{n} y_t \cdot (\boldsymbol{w}[i_t] - \boldsymbol{w}[j_t]) + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \tag{7}$$

$$\leq \inf_{\boldsymbol{w} \in \mathbb{R}^m : \max_{i,j} \boldsymbol{w}[i] - \boldsymbol{w}[j] \leq B} \frac{1}{n}\sum_{t=1}^{n} y_t \cdot (\boldsymbol{w}[i_t] - \boldsymbol{w}[j_t]) + \frac{B}{2}\sqrt{\frac{m}{n}} \tag{8}$$

This benchmark satisfies the property that $\mathbb{E}\left[\phi(\boldsymbol{\varepsilon})\right] \geq 0$. *This is because*

$$\begin{aligned}
\mathbb{E}\left[\phi(\boldsymbol{\varepsilon})\right] &= \mathbb{E}\left[\inf_{\boldsymbol{w}\in\mathbb{R}^m : \max_{i,j} \boldsymbol{w}[i]-\boldsymbol{w}[j]\leq B} \frac{1}{n}\sum_{t=1}^{n} y_t \cdot (\boldsymbol{w}[i_t] - \boldsymbol{w}[j_t])\right] + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \\
&= \mathbb{E}\left[\inf_{\boldsymbol{w}\in[0,B]^m} \frac{1}{n}\sum_{t=1}^{n} \epsilon_t(\boldsymbol{w}[i_t] - \boldsymbol{w}[j_t])\right] + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{m} \min_{\boldsymbol{w}[i]\in[0,B]} \sum_{t=1}^{n} \boldsymbol{w}[i]\epsilon_t\left(\mathbb{1}_{\{i_t=i\}} - \mathbb{1}_{\{j_t=i\}}\right)\right] + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{m} \min\left\{B\sum_{t=1}^{n}\epsilon_t\left(\mathbb{1}_{\{i_t=i\}} - \mathbb{1}_{\{j_t=i\}}\right), 0\right\}\right] + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \\
&= \frac{B}{n}\sum_{i=1}^{m}\mathbb{E}\left[\min\left\{\sum_{j=1}^{n_i}\epsilon_j, 0\right\}\right] + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \\
&\geq -\frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} + \frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} = 0
\end{aligned}$$

where in the last line we used the fact that for any integer N, $\mathbb{E}\left[\min\left\{\sum_{j=1}^{N}\epsilon_j, 0\right\}\right] \geq -\sqrt{N}/2$. *Hence, from Lemma 3 this benchmark is achievable by the gambler using the strategy* $\widehat{y}_t = n \cdot \mathbb{E}[\phi(y_{1:t-1}, -1, \varepsilon_{t+1:n}) - \phi(y_{1:t-1}, +1, \varepsilon_{t+1:n})]$. *Finally, noting that square-root is a concave function and applying Jensen's inequality, yields that* $\frac{B}{2n}\sum_{i=1}^{m}\sqrt{n_i} \leq \frac{B}{2}\sqrt{\frac{m}{n}}$.