CS667 **Lecture 9: Volume Rendering Equation** 22 February 2005

Victor Wang Lecturer: Steve Marschner

1 Introduction

Last lecture we introduced the scattering equation:

$$(\omega \cdot \nabla)L(x,\omega) + \sigma_t(x)L(x,\omega) = \epsilon(x,\omega) + \sigma_s(x)\int_{4\pi} p(x,\omega,\omega')L(x,\omega')d\omega'$$

This equation is difficult to evaluate because it contains both derivatives and integrals. In order to solve it using Monte Carlo integration, we must first rewrite it in a form that contains only integrals (When all you have is a hammer, everything needs to be rewritten as a nail). Rewriting the scattering equation requires a basic understanding of *Ordinary Differential Equations*(ODEs).

2 Brief ODE Review

An ordinary differential equation is an equality that consists of a function and its derivatives. An ODE is ordinary because all functions and derivatives in the equality are with respect to a *single* independent variable. We will focus our review of ODEs to a few simple types.

2.1 Linear Homogeneous 1^{st}-order ODEs with constant coefficients

A Linear Homogeneous 1^{st}-order ODE with constant coefficients can be written as :

$$y'(x) + \alpha y(x) = 0$$

This ODE is homogeneous because the right hand side of the equation is equal to zero. It is of the 1^{st}-order because the 1^{st} derivative is the highest derivative that appears in the equality(As opposed to a 2^{nd}-order ODE which would have the 2^{nd} derivative as the highest derivative in the equality. i.e. $y'' + y' + y = 0$). We can solve this ODE for $y(x)$ by equating the integrals of the two sides:

$$\begin{aligned} y' + \alpha y &= 0 \\ \frac{y'}{y} &= -\alpha \\ \int \frac{y'}{y}dx &= -\int \alpha\, dx \\ \ln y + C &= -\alpha x \\ y(x) &= Ce^{-\alpha x} \end{aligned}$$

2.2 Linear 1^{st} order homogeneous ODE

By allowing coefficients to vary with respect to the independent variable, we define a Linear 1^{st} order homogeneous ODE. It can be written as:

$$y'(x) + p(x)y(x) = 0$$

Solving for $y(x)$ in the same way as for the constant case:

$$\begin{aligned}
y' + py &= 0 \\
\frac{y'}{y} &= -p \\
\int \frac{y'}{y} dx &= -\int p\, dx \\
\ln y &= -\int p\, dx + C \\
y(x) &= Ce^{\int^x p(x) dx}
\end{aligned}$$

2.3 Linear 1^{st} order ODE

Finally, by relaxing the homogeneous constraint we can write a general Linear 1^{st} order ODE as:

$$y'(x) + p(x)y(x) = q(x)$$

A trick that we can employ to solve this equation is by writing the equation as a product of functions:

$$(\mu y)' = g$$

We can integrate the product of functions to get $y(x)$:

$$\begin{aligned}
\int (\mu y)' dx &= \int g\, dx \\
(\mu y) &= \int g\, dx + C \\
y(x) &= \frac{\int^x g(x) dx + C}{\mu(x)}
\end{aligned}$$

Next, make $(\mu y)' = g$ look like $y' + py = q$:

$$\begin{aligned}
(\mu y)' &= g \\
\mu y' + \mu' y &= g \\
y' + \frac{\mu'}{\mu} y &= \frac{g}{\mu}
\end{aligned}$$

Then use "pattern matching" to identify μ and g:

$$\begin{aligned}
p = \frac{\mu'}{\mu} &\Rightarrow \mu = e^{\int p} \\
q = \mu g &\Rightarrow g = q e^{\int p}
\end{aligned}$$

By substituting in the above equations, we can write $y(x)$ as:

$$\begin{aligned}
y(x) &= \frac{\int^x q(t) e^{\int^t p(t') dt'} dt + C}{e^{\int^x p(t) dt}} \\
&= \int^x q(t) e^{\int^t p(t') dt'} dt \cdot e^{-\int^x p(t') dt'} + Ce^{-\int^x p(t) dt} \\
&= \int^x q(t) e^{-[\int^x p(t') dt' - \int^t p(t') dt']} dt + Ce^{-\int^x p(t) dt} \\
&= \int^x q(t) e^{-\int_t^x p(t') dt'} dt + Ce^{-\int^x p(t) dt}
\end{aligned}$$

3 Derivation of the Volume Rendering Equation

Recall that the scattering equation is:

$$(\omega \cdot \nabla)L(x,\omega) + \sigma_t(x)L(x,\omega) = \epsilon(x,\omega) + \sigma_s(x)\int_{4\pi} p(x,\omega,\omega')L(x,\omega')d\omega'$$

Let's focus on just computing the vlaue of $L(x,\omega_0)$ for a particular point x and direction ω_0. In our discussion rays originate at some arbitrary surface point y (Not to be confused with the function y in the ODE equations). Thus, $L(y,\omega) = L_e(y,\omega)$. This means that the radiance arriving at x is directly dependent on *only* what happens along the ray through x in the direction of $-\omega_0$. An illustrated example is provided in the figure below:

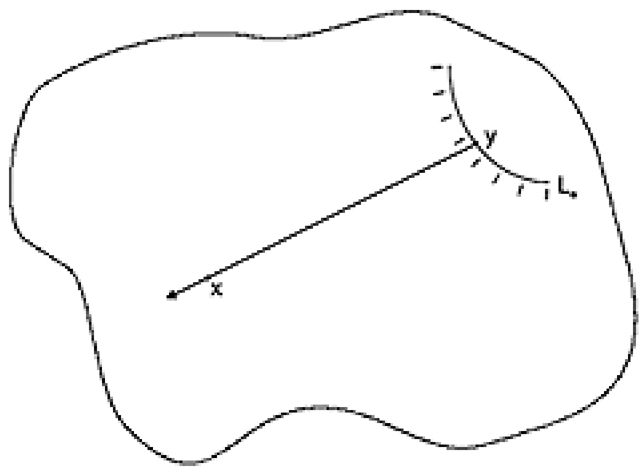

Figure 1: Restricting the solution of the scattering equation to a simpler domain, the ray between x and y.

First we can parameterize x as the distance t from a point y in the direction ω_0. Since the direction of the ray, ω_0, is fixed, we can reduce the functions in the scattering equation to smaller functions of t.

$$\begin{aligned} x(t) &= y + t\omega_0 \\ L(t,\omega') &= L(x(t),\omega') \\ \sigma_t(t) &= \sigma_t(x(t)) \\ \sigma_s(t) &= \sigma_s(x(t)) \\ \sigma_a(t) &= \sigma_a(x(t)) \\ \epsilon(t) &= \epsilon(x(t)) \\ p(t,\omega') &= p(x(t),\omega_0,\omega') \end{aligned}$$

The scattering equation can then be written as:

$$\underbrace{L'(t)}_{y'} + \underbrace{\sigma_t(t)}_{p}\underbrace{L(t)}_{y} = \underbrace{\epsilon(t) + \sigma_s(t)\int_{4\pi} p(t,\omega')L(t,\omega')d\omega'}_{q}$$

which is a Linear 1^{st} order ODE of the form:

$$y'(x) + p(x)y(x) = q(x)$$

3.1 Simple Case: Just Absorbtion($\epsilon = 0$, $\sigma_s = 0$)

If the medium neither emits or scatters light, the scattering equation takes on the form a Homogeneous Linear 1^{st}-order ODE:

$$\underbrace{L'(t)}_{y'} + \underbrace{\sigma_t(t)}_{p} \underbrace{L(t)}_{y} = 0$$

Solving for $L(t)$ yields the result:

$$L(t) = Ce^{-\int^t \sigma_a(t)dt}$$

From the boundary conditions $L(y, \omega) = L_e(y, \omega)$. This defines the constant C.

$$L(0) = L_e(y, \omega_0) = C$$

The final homogeneous solution becomes:

$$L(t) = L_e(y, \omega_0)e^{-\int_0^t \sigma_a(t')dt'}$$

We can then write the solution in 3D form [1] [2]:

$$L(x, \omega) = L_e(\Psi(x, -\omega), \omega)e^{-\int_0^{||x-y||} \sigma_a(\Psi(x,\omega)+t'\omega)dt'}$$
$$L(x, \omega) = L_e(y, \omega)e^{-\int_y^x \sigma_a(x')dx'}$$

This shows that radiance decays exponentially, and we get the total attenuation by integrating σ_a along the line of sight. This attenuation factor is important enough that we'll define it as:

$$\alpha(x, y) = e^{-\int_y^x \sigma_a(x')dx'} = \alpha(y, x)$$

If we are rendering a ray traced image, one integral per ray is all we need to compute to incorporate a purely absorbing medium.

3.2 Absorbtion/Emission ($\sigma_s = 0$):

Now we will assume that the medium is both emissive and absorbing.

$$\underbrace{L'(t)}_{y'} + \underbrace{\sigma_t(t)}_{p} \underbrace{L(t)}_{y} = \underbrace{\epsilon(t)}_{q}$$

We can solve this equation like any Linear 1^{st}-order ODE. Again, to get bounds and constraints, observe that $L(0)$ is $L_e(y, \omega_0)$ resulting in $C = L_e(y, \omega_0)$:

$$L(t) = \int^t \epsilon(t')e^{-\int_{t'}^t \sigma_a(t'')dt''}dt' + Ce^{-\int^t \sigma_a(t')dt'}$$
$$L(t) = \int_0^t \epsilon(t')e^{-\int_{t'}^t \sigma_a}dt' + L(y, \omega_0)e^{-\int_0^t \sigma_a}$$

Physically this means: As we are collecting light emitted along the ray, light coming from each point is attenuated by the medium in front of it. Rewriting the above equation in 3D form yields the result:

$$L(x, \omega) = \int_x^y \alpha(x', x)\epsilon(x', \omega)dx' + \alpha(y, x)L_e(y, w)$$

[1] The $\Psi(x, \omega)$ funtion is the ray-casting function

[2] Note the abbreviated notation $\int_y^x f(x')\,dx'$ for an integral along the line segment from y to x

3.3 Full Volume Rendering Equation

We are now ready to tackle a medium that scatters, emits, and absorbs light. This medium can be described as:

$$\underbrace{L'(t)}_{y'} + \underbrace{\sigma_t(t)}_{p}\underbrace{L(t)}_{y} = \underbrace{\epsilon(t) + \sigma_s(t)\int_{4\pi} p(t,\omega')L(t,\omega')d\omega'}_{q}$$

Instead of solving this equation in 1D (which is similar to the previous examples), let's skip ahead to the 3D solution:

$$L(x,\omega) = \int_y^x \alpha(x',x)\left(\epsilon(x',\omega) + \sigma_s(x')\int_{4\pi} p(x',\omega',\omega)L(x',\omega')d\omega'\right)dx' + \alpha(x,y)L_e(y,\omega)$$

Rearranging the terms a little we can seperate $L(x,\omega)$ into a "source" term that doesn't depend on $L(x,\omega)$, and a "scattering" term that *does* depend on $L(x,\omega)$.

$$L(x,\omega) = \underbrace{\int_y^x \alpha(x',x)\epsilon(x',\omega)dx' + \alpha(y,x)L_e(y,\omega)}_{source} + \underbrace{\int_y^x \alpha(x',x)\sigma_s(x')\left(\int_{4\pi} p(x',\omega',\omega)L(x',\omega')d\omega'\right)dx'}_{scattering}$$

Recall the Vacuum Rendering Equation had the form:

$$L_e(x,\omega) = L_e^0(x,\omega) + \int_\pi f_r L_e d\mu$$

As it turns out, the Volume Rendering Equation can be expressed in an analogous intergral structure:
Let,

$$L^0(x,\omega) = \alpha(y,x)L_e(y,\omega) + \int_y^x \alpha(x',x)\epsilon(x')dx'$$
$$K(x,x',\omega,\omega') = \alpha(x',x)\sigma_s(x')p(x',\omega',\omega)$$

It follows that:

$$L(x,\omega) = L^0(x,\omega) + \int_y^x \int_{4\pi} K(x,x',\omega,\omega')L(x',\omega')d\omega' dx'$$