CS667 **Lecture 3: The BRDF, Light Transport in a Vacuum** 3 February 2004

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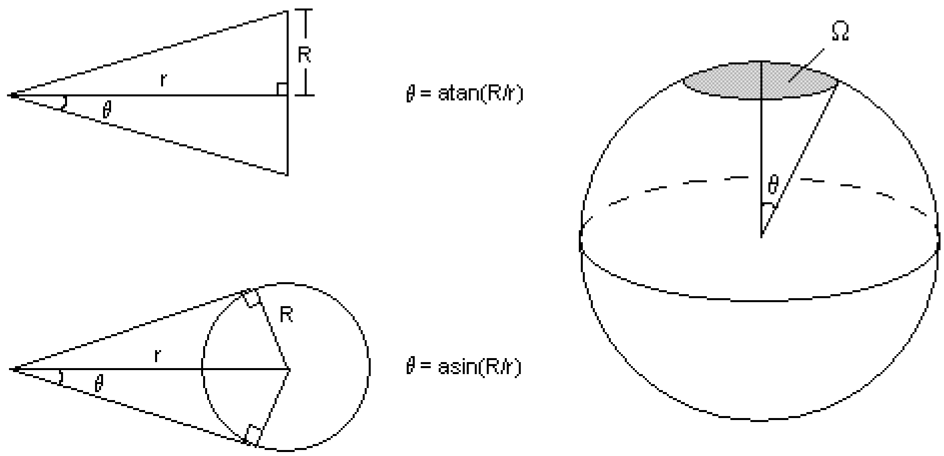
# 1 Radiometry examples

To conclude discussion of radiometry, we present three example exercises.

## 1.1 Solid angle of a disc or sphere

Given a disc or sphere of radius $R$ with center at distance $r$ from the point of interest, what is the solid angle it subtends?




The solid angle of this region is equal to the area of the cap of the unit sphere, $\Omega$. If we define the measure $\sigma$ as

$$\sigma(\Omega) = |\Omega|$$

then the area is

$$\int_{\Omega} d\sigma = \int_0^{2\pi} \int_0^{\theta} sin\ \theta\ d\theta\ d\phi$$

Note that $\theta$ is the angle down from the top of the sphere, and $\phi$ is the angle in the equatorial plane, measured counterclockwise. This will be the convention used for the duration of this course. We evaluate the integral as follows:

$$= 2\pi \int_0^{\theta} sin\ \theta\ d\theta = 2\pi[-cos\ \theta - (-cos\ 0)] = 2\pi(1 - cos\ \theta)$$

Note that the quantity $(1 - cos\ \theta)$ is the height of the spherical cap, and that this result generalizes for arbitrary bands around the sphere:

$$\text{Area of band} = 2\pi(cos\ \theta_1 - cos\ \theta_2)$$

## 1.2 Radiance of the sun, approximately

This example gives an idea of the magnitude of radiance values in the real world. Consider the irradiance of the sun on a flat surface at noon.

- known irradiance of the sun: 500 $W/m^2$
- known angular subtense: $1/2^o$ or 1/100 radian

We estimate the solid angle by treating the spherical cap as a disc, with diameter 1/100 on the unit sphere and thus has area $\pi/40000$ steradian. Since the illumination is perpendicular, this is also the projected solid angle. Since radiance is irradiance per unit projected solid angle, we find

$$L = \frac{500 \text{ W/m}^2}{\pi/40000 \text{ sr}} \approx 6 \times 10^6 \text{ W/m}^2\text{sr}$$

### 1.3 Reflection from a Lambertian reflector

A Lambertian reflector reflects a fraction $R$ of its incident flux, emitting it uniformly in all directions. That is,

$$\text{(radiant exitance) } M = R * E \text{ (irradiance)}$$

Recall also that

$$M(\mathbf{x}) = \int_{\mathbb{H}^2} L(\mathbf{x}, \omega)\, d\mu(\omega) = \int_{\mathbb{H}^2} L\, d\mu = L \int_{\mathbb{H}^2} d\mu = \pi L$$

where $\mu$ is the projected solid angle measure. Combining these two results yields

$$\pi L = R * E \quad \text{and so} \quad \mathrm{L} = \frac{\mathrm{R}}{\pi}\mathrm{E}$$

## 2 The Bidirectional Reflectance Distribution Function (BRDF)

### 2.1 Definition

Surface reflection is an operator, taking as input an incident radiance distribution $L_i$ and producing a reflected radiance distribution $L_e$ as output. That is, $L_e = \mathcal{R}(L_i)$.

### 2.2 Linearity of the BRDF

A key property of $\mathcal{R}$ is linearity: $\mathcal{R}(A + B) = \mathcal{R}(A) + \mathcal{R}(B)$ This linearity allows us to treat a radiance distribution $A$ as a sum of small light sources $A_j$, each contributing radiance $L_j$ from solid angle $\Omega_j$ around $\omega_j$, and have $\mathcal{R}(A) = \sum_j \mathcal{R}(A_j)$

This means that to predict the reflection of any radiance distribution, we only need to know the reflection for small sources. This is exactly what the BRDF tells us: the reflected distribution from a small source. We can define the BRDF $f_r$ as the exitant radiance in a direction per incident radiance from a direction per unit projected solid angle. That is,

$$f_r(\omega_i, \omega_r) = \frac{L_r}{L_i}/\mu(\Omega_i)$$

Equivalently,

$$L_r = f_r(\omega_i, \omega_r)\; L_i\; \mu(\Omega_i)$$

For our sum of small light sources $A_j$, we have

$$\mathcal{R}(A)(\omega_r) = \sum_j f_r(\omega_j, \omega_r)\; L_j\; \mu(\Omega_j)$$

Or as the limit as $\Omega_j$ gets small:

$$L_r(\omega_r) = \int_{\mathbb{H}^2} f_r(\omega_i, \omega_r)\; L_i(\omega_i)\; d\mu(\omega_i)$$

Two other ways to think about the BRDF are:

- $f_r(\cdot, \omega_r)$ represents the "sensitivity" to radiance per unit projected solid angle
- $f_r(\omega_i, \cdot)$ represents the reflected radiance for a collimated incident beam.

### 2.3 Properties of the BRDF

It should be obvious that a BRDF needs to conserve energy: the flux leaving a surface (radiant exitance) must be $\leq$ the flux incident on the surface (irradiance) for all incident distributions:

$$\int_{\mathbb{H}^2} L_r\; d\mu \leq \int_{\mathbb{H}^2} L_i\; d\mu$$

This is true if and only if it holds for collimated illumination:

$$\int_{\mathbb{H}^2} f_r(\omega_i, \omega_r)\; d\mu(\omega_r) \leq 1 \tag{1}$$

The forward implication is obvious, and the reverse implication is shown via integration:

$$M = \int_{\mathbb{H}^2} L_r\; d\mu = \int_{\mathbb{H}^2} \int_{\mathbb{H}^2} f_r(\omega_i, \omega_r)\; L_i(\omega_i)\; d\mu(\omega_i)\; d\mu(\omega_r)$$

Swapping the order of integration, we have

$$\int_{\mathbb{H}^2} L_i(\omega_i) \underline{\int_{\mathbb{H}^2} f_r(\omega_i, \omega_r)\; d\mu(\omega_r)}\; d\mu(\omega_i)$$

By (1), the underlined integral must evaluate to $\leq 1$. Thus

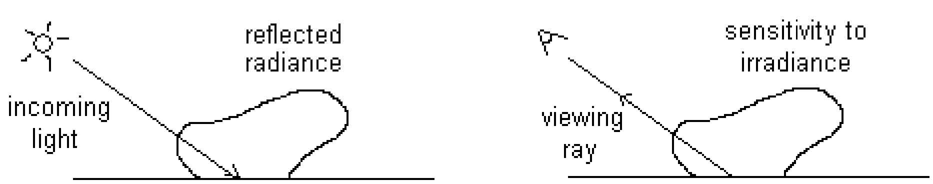
$$M \leq \int_{\mathbb{H}^2} L_i(\omega_i)\; d\mu(\omega_i) = E$$

as claimed.

A less obvious property is Helmholtz reciprocity, which states that the BRDF has a symmetry with respect to swapping its arguments:

$$f_r(\omega_i, \omega_r) = f_r(\omega_r, \omega_i)$$

The physical interpretation for this reciprocity is that the sensitivity distribution looks like the radiance distribution:




This is a very important property, and is fundamental to many rendering algorithms.

## 3 Light Transport in a vacuum

Consider the transport of light through a vacuum, by which we mean there is no participating medium. Take the following as ground rules:

- The scene is composed of surfaces floating in a vacuum. Let all the surfaces considered together be a piecewise smooth surface (a 2-manifold) $\mathcal{M}$.
- Reflection occurs pointwise, as all surfaces are opaque and obey valid BRDFs.
- The output we are interested in - the camera image - is just a set of averages over the light reflected from the scene surfaces, with one measurement made per pixel.
- There is an enclosure surrounding all of $\mathcal{M}$, to avoid special cases for the background.
- All light in the scene is initially emitted from the surfaces

Also define:

- $L_e(\mathbf{x}, \omega_e)$ is the exitant radiance from point $\mathbf{x} \in \mathcal{M}$ to direction $\omega_e$.
- $L_e : \mathcal{M} \times \mathbb{H}^2 \rightarrow \mathbb{R}$
- $L_i(\mathbf{x}, \omega_i)$ is the incident radiance on point $\mathbf{x} \in \mathcal{M}$ from direction $\omega_i$
- $L_i : \mathcal{M} \times \mathbb{H}^2 \rightarrow \mathbb{R}$
  note that $\omega$ always faces away from the surface!
- $f_r(\mathbf{x}, \omega_i, \omega_e)$ is the BRDF at point $\mathbf{x}$
- $f_r : \mathcal{M} \times \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow \mathbb{R}$

From all this, the BRDF definition gives:

$$L_e(\mathbf{x}, \omega_e) = \int_{\mathbb{H}^2} f_r(\mathbf{x}, \omega_i, \omega_e)\; L_i(\mathbf{x}, \omega_i)\; d\mu(\omega_i)$$

or

$$L_e = \mathbf{K} L_i \quad \text{where } \mathbf{K} \text{ is the reflection operator}$$

We can think of $\mathbf{K}$ as the whole surface reflectance for all points everywhere rolled into a single linear operator. We also include emittance, which adds to the reflection:

$$L_e = \mathbf{K} L_i + L_e^0$$

Where $L_e^0(\mathbf{x}, \omega_e)$ is the radiance emitted from point $\mathbf{x}$ in direction $\omega_e$.

At this point, this is just a restatement of surface reflection. To make a solvable equation we need to relate $L_i$ to $L_e$. Fortunately, because we are considering light transport in a vacuum, they are the same function - only with permuted domains. That is, $L_i(\mathbf{x}, \omega) = L_e(\mathbf{y}, -\omega)$ for the point $\mathbf{y}$ that is visible from $\mathbf{x}$ when looking in the direction $\omega$. This is ray casting, essentially.

We can then define a transport operator $\mathbf{G}$ such that $L_i = \mathbf{G} L_e$:

$$(\mathbf{G} L_e)(\mathbf{x}, \omega) = L_e(\psi(\mathbf{x}, \omega), -\omega)$$

Where $\psi$ is the ray casting function, with $\psi(\mathbf{x}, \omega) = \mathbf{y}$, and $\psi : \mathcal{M} \times \mathbb{H}^2 \rightarrow \mathcal{M}$

Finally, we can substitute this into our surface reflection equation, resulting in

$$L_e = \mathbf{K}\mathbf{G} L_e + L_e^0$$

This is a very compact way to write down the rendering problem and to expose the algebraic structure. As a final note, let $\mathbf{1}$ be the identity operator. Then we have

$$\mathbf{1}L_e - \mathbf{KG}L_e = L_e^0$$
$$L_e = (\mathbf{1} - \mathbf{KG})^{-1}L_e^0$$
$$L_e = L_e^0 + \mathbf{KG}(L_e^0 + \mathbf{KG}(L_e^0 + ...))$$

Which is an intuitive representation for recursive ray tracing.

Next lecture we will examine Kajiya's formulation of the rendering equation using areas.