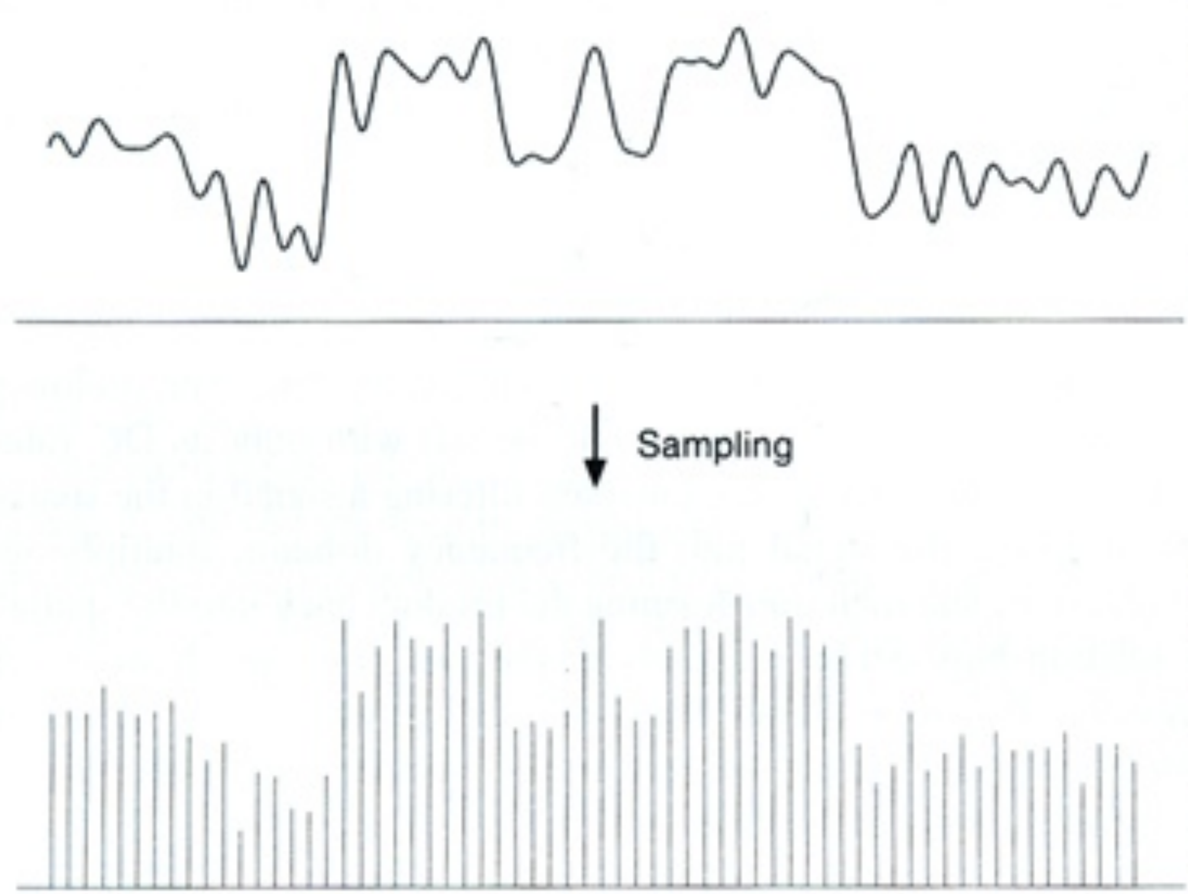
CS6640 Computational Photography

11. Sampling theory

Sampled representations

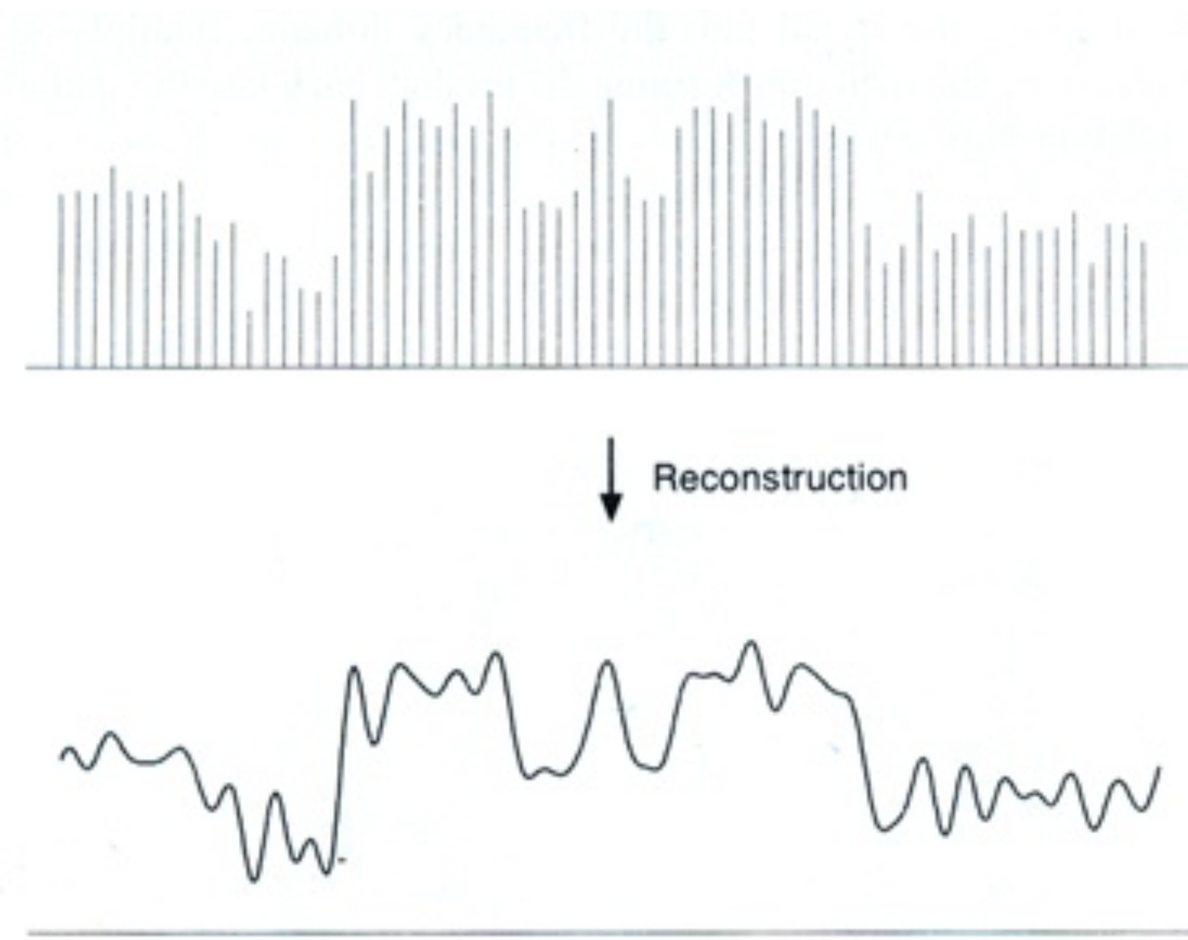
- How to store and compute with continuous functions?
- Common scheme for representation: samples
  - write down the function's values at many points



[FvDFH fig.14.14b / Wolberg]

Reconstruction

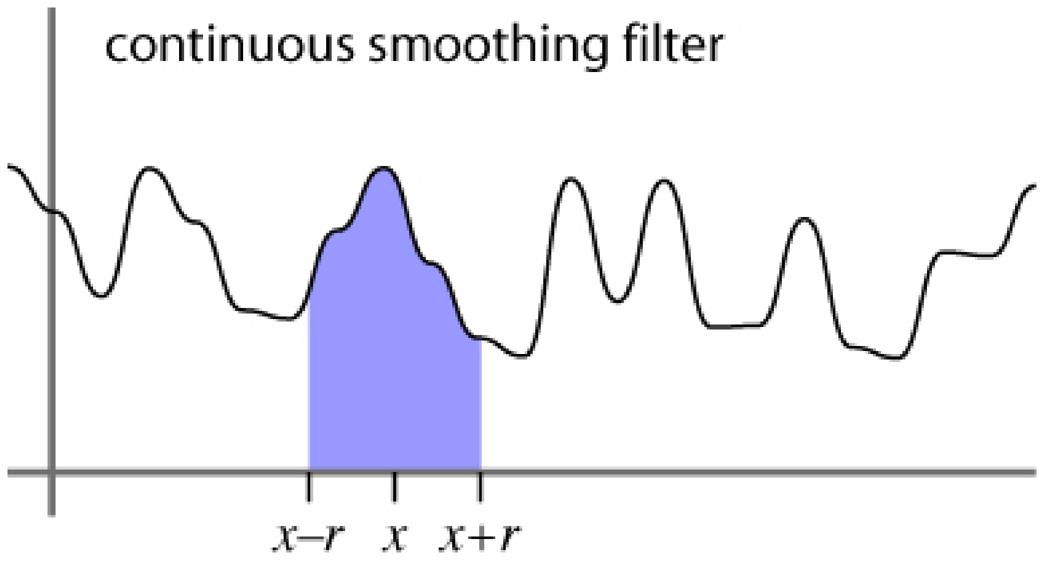
- Making samples back into a continuous function
  - for output (need realizable method)
  - for analysis or processing (need mathematical method)
  - amounts to “guessing” what the function did in between

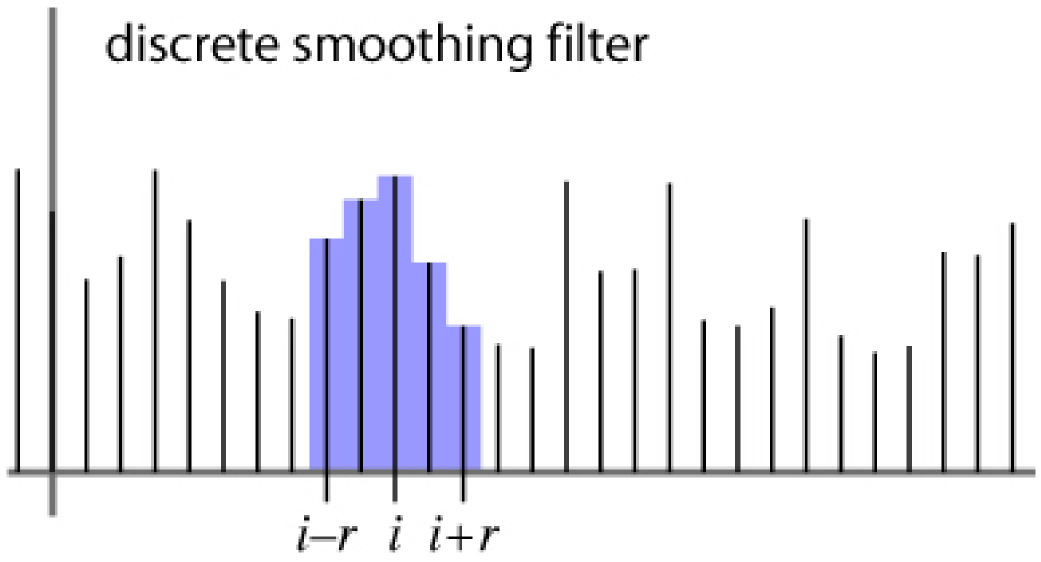


[FvDFH fig.14.14b / Wolberg]

Filtering

- Processing done on a function
  - can be executed in continuous form (e.g. analog circuit)
  - but can also be executed using sampled representation
- Simple example: smoothing by averaging



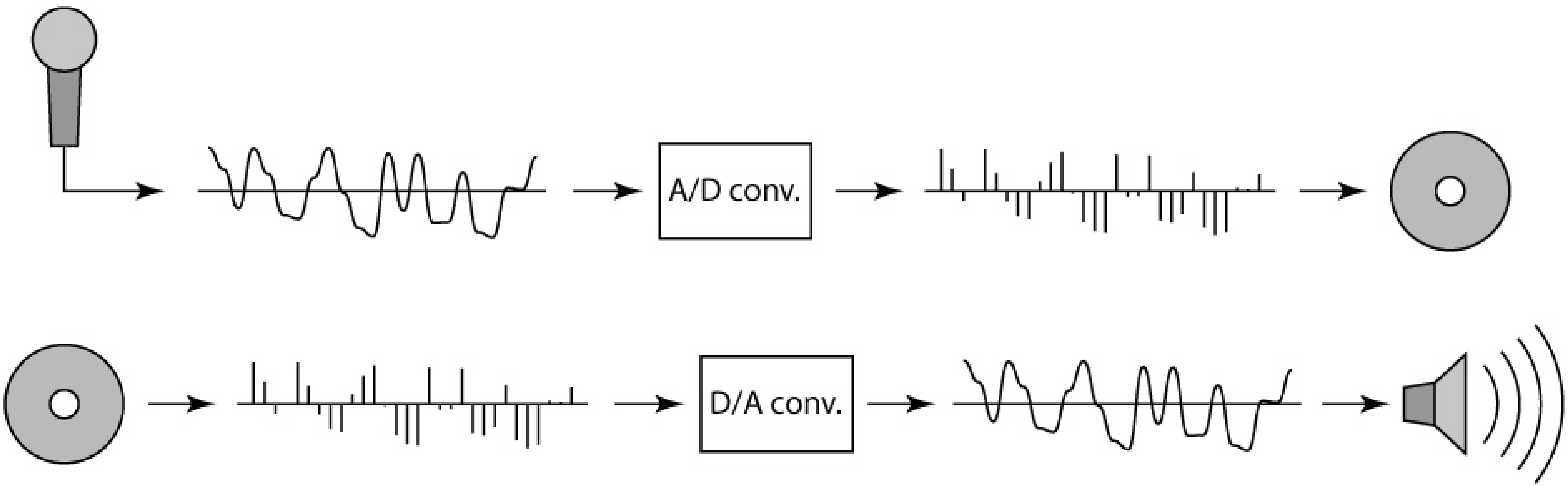


Roots of sampling

- Nyquist 1928; Shannon 1949
  famous results in information theory
- 1940s: first practical uses in telecommunications
- 1960s: first digital audio systems
- 1970s: commercialization of digital audio
- 1982: introduction of the Compact Disc
  the first high-profile consumer application
- This is why all the terminology has a communications or audio “flavor”
  early applications are 1D; for us 2D (images) is important

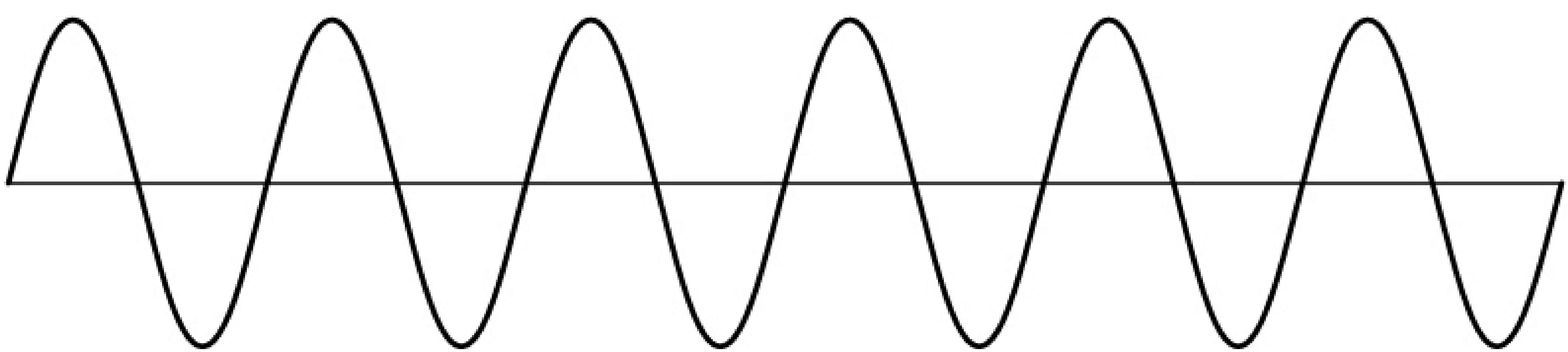
Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
  - how can we be sure we are filling in the gaps correctly?



Undersampling

- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency
  - also was always indistinguishable from higher frequencies
  - *aliasing*: signals "traveling in disguise" as other frequencies



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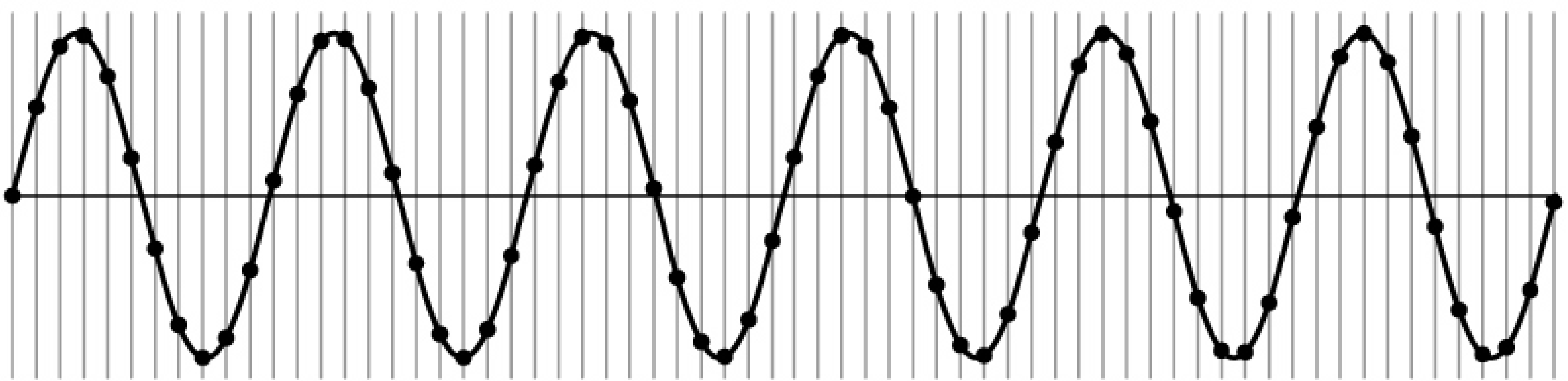

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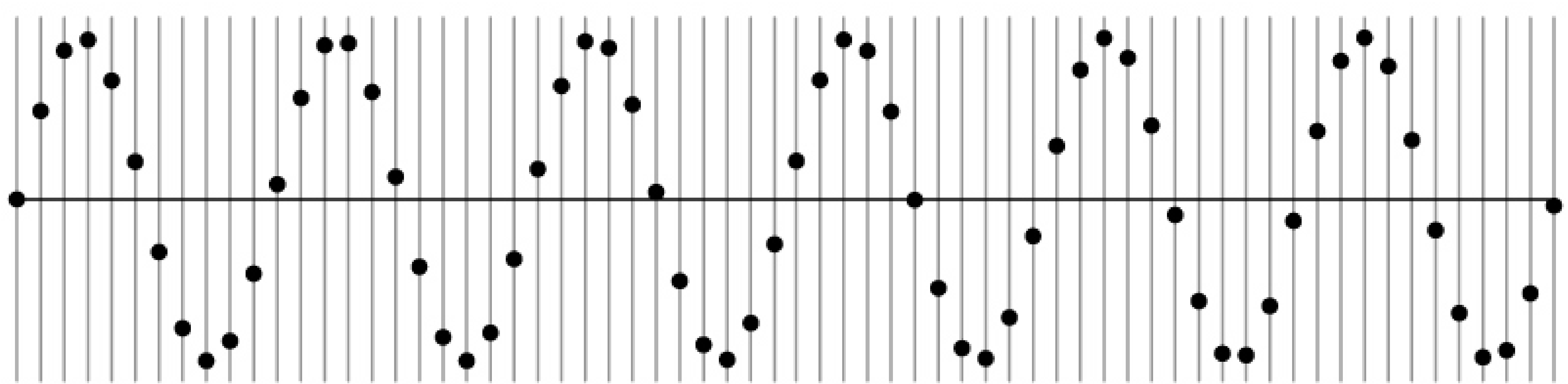

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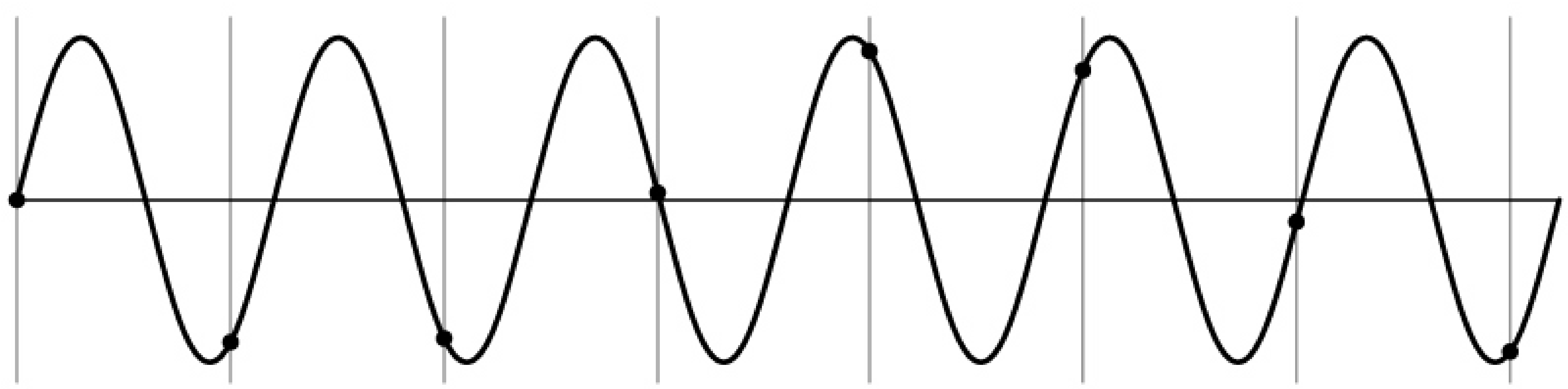

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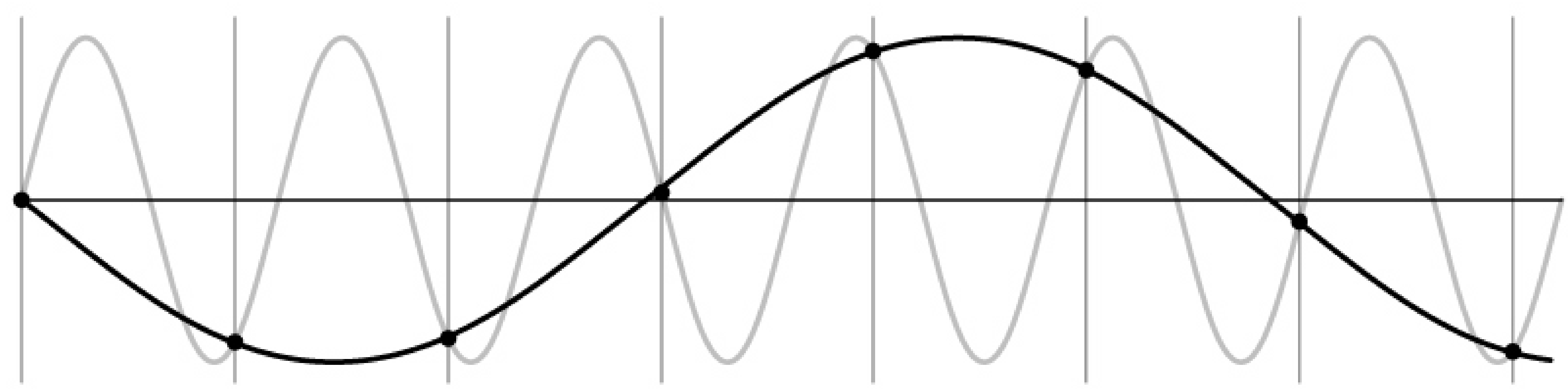

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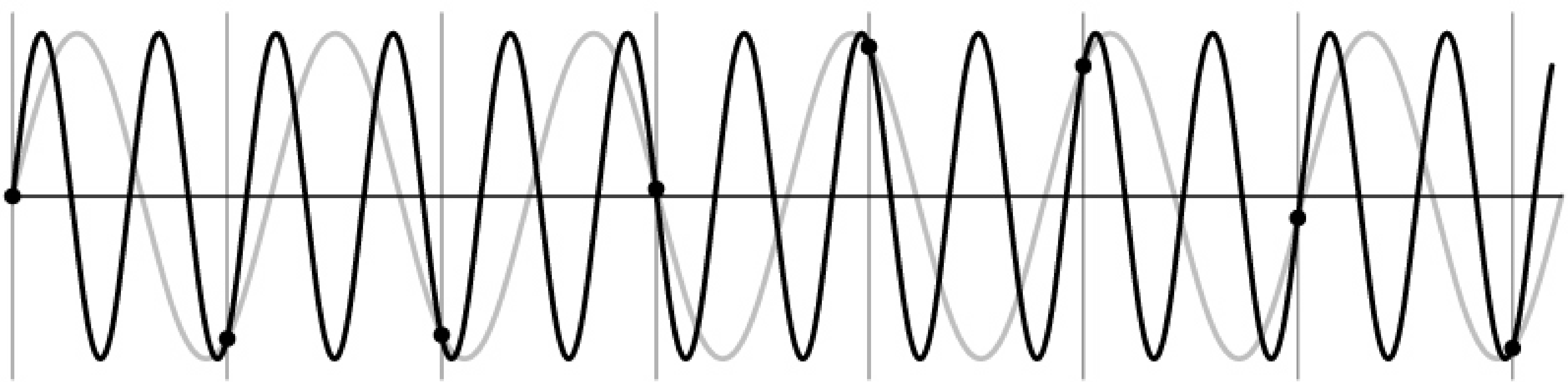
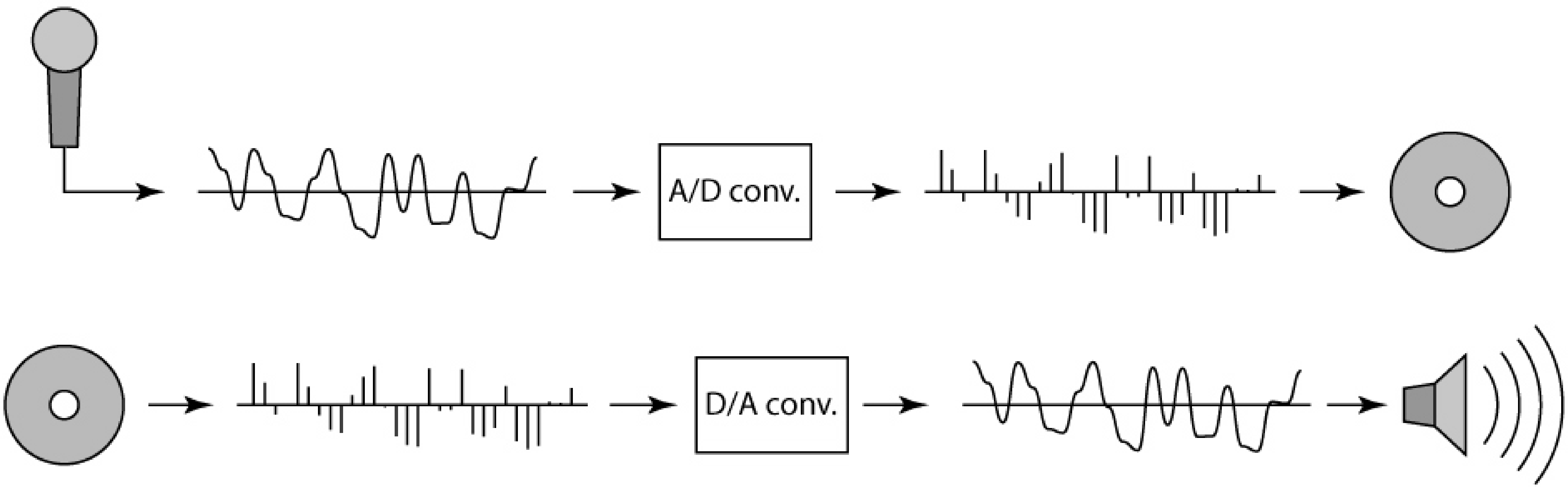


Sneak preview

- Sampling creates copies of the signal at higher frequencies
- Aliasing is these frequencies leaking into the reconstructed signal
  - frequency $f_s - x$ shows up as frequency x
- The solution is filtering
  - during sampling, filter to keep the high frequencies out so they don't create aliases at the lower frequencies
  - during reconstruction, again filter high frequencies to avoid including high-frequency aliases in the output.

Preventing aliasing

- Introduce lowpass filters:
  - remove high frequencies leaving only safe, low frequencies
  - choose lowest frequency in reconstruction (disambiguate)



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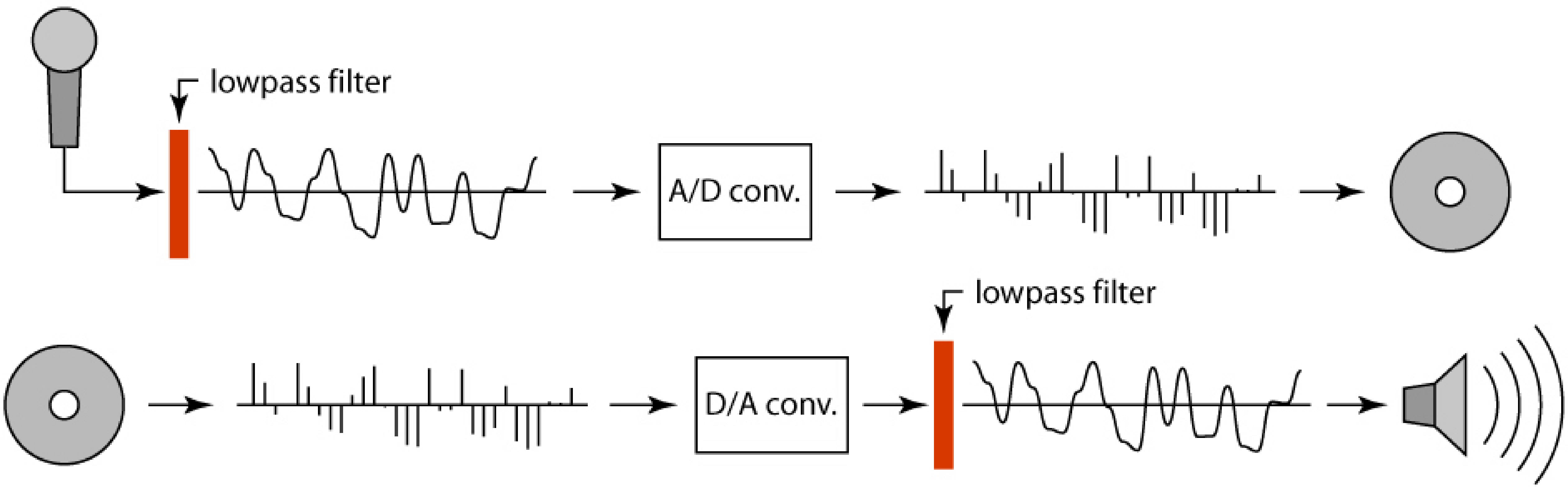
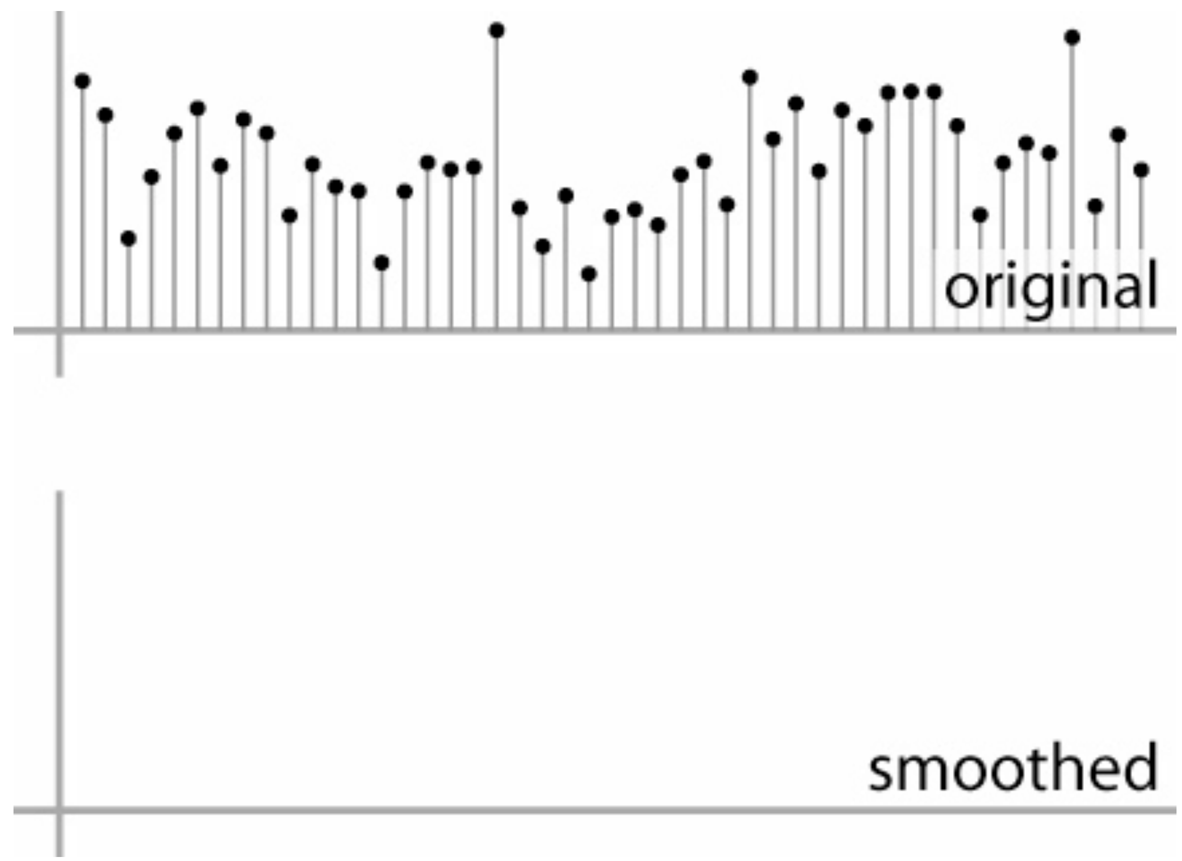


Linear filtering: a key idea

- Transformations on signals; e.g.:
  - bass/treble controls on stereo
  - blurring/sharpening operations in image editing
  - smoothing/noise reduction in tracking
- Key properties
  - linearity: filter(f + g) = filter(f) + filter(g)
  - shift invariance: behavior invariant to shifting the input
    - delaying an audio signal
    - sliding an image around
- Can be modeled mathematically by *convolution*

Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



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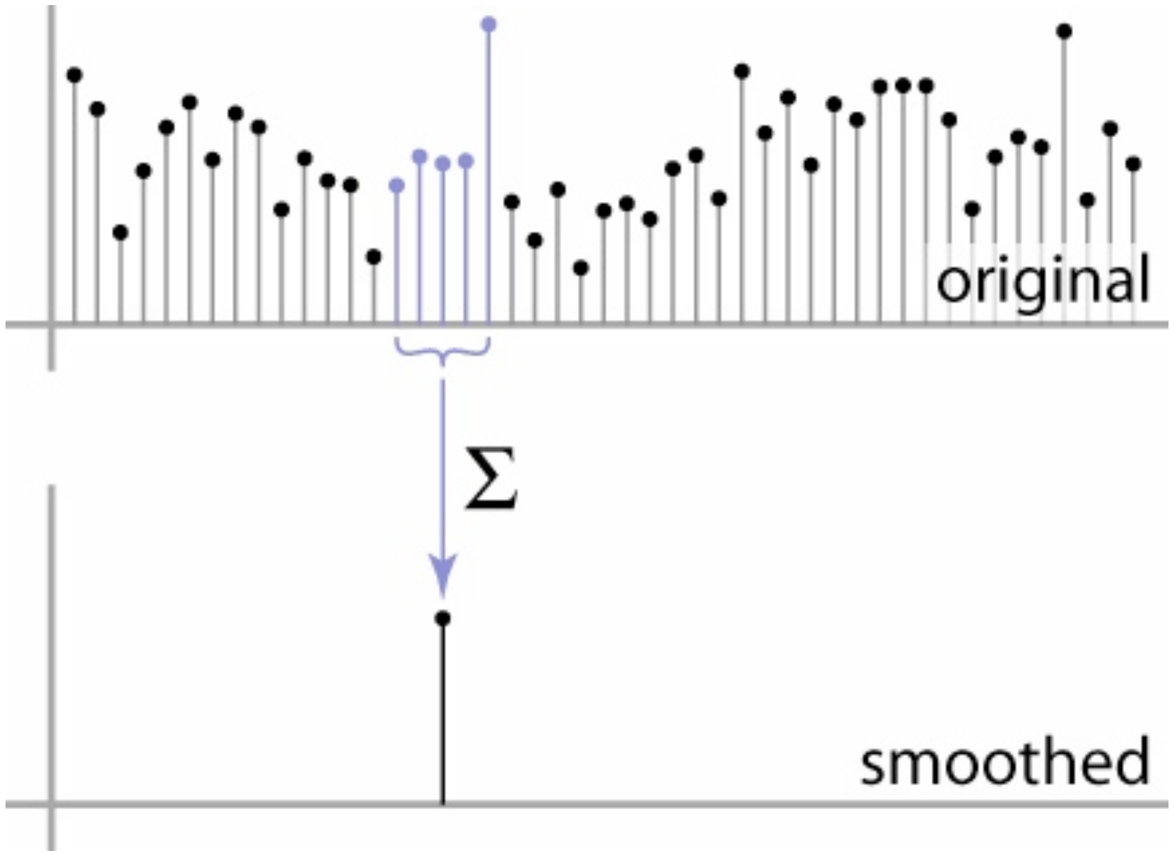

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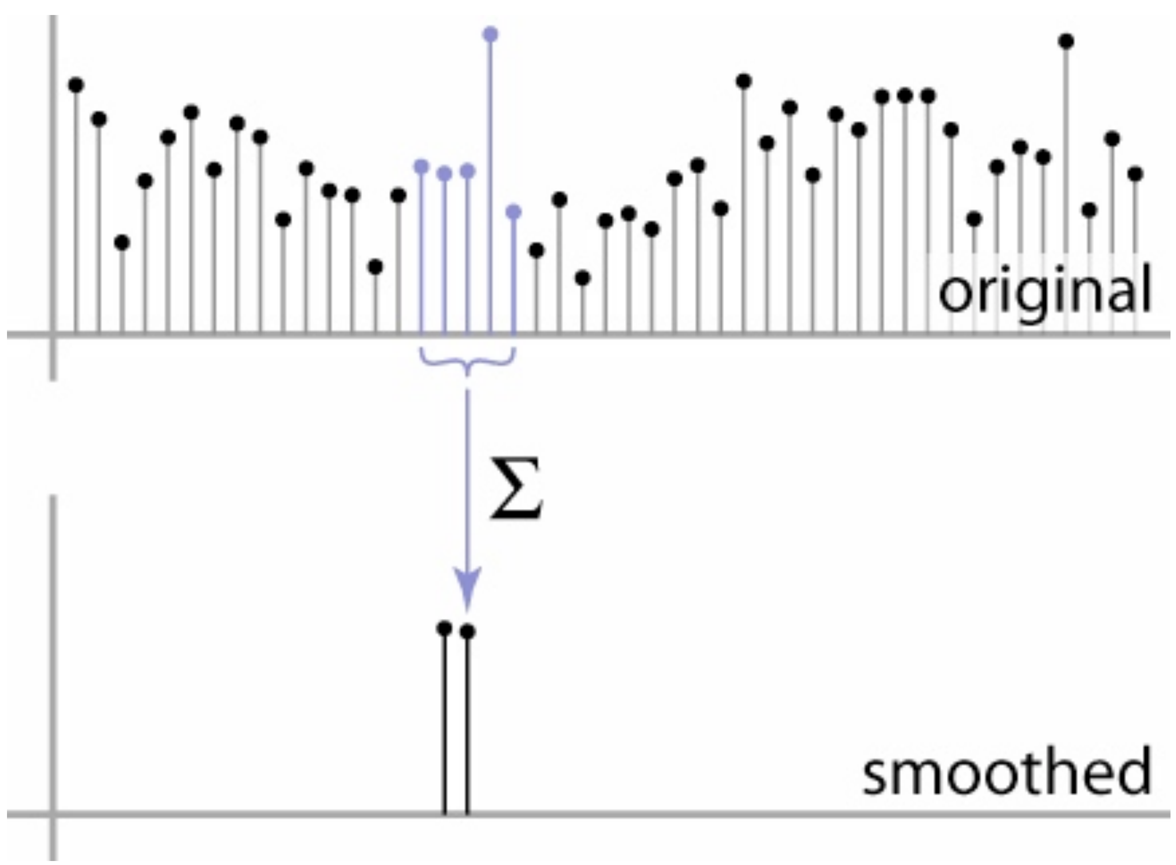

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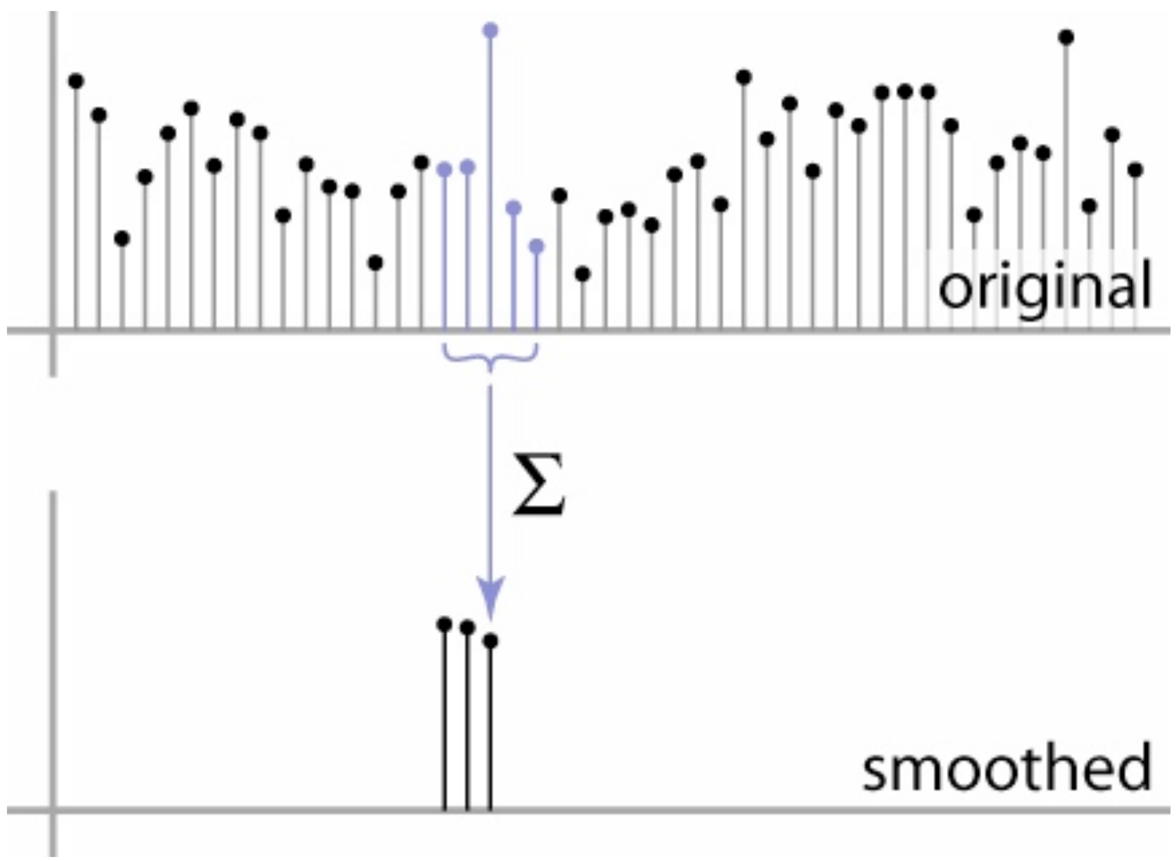

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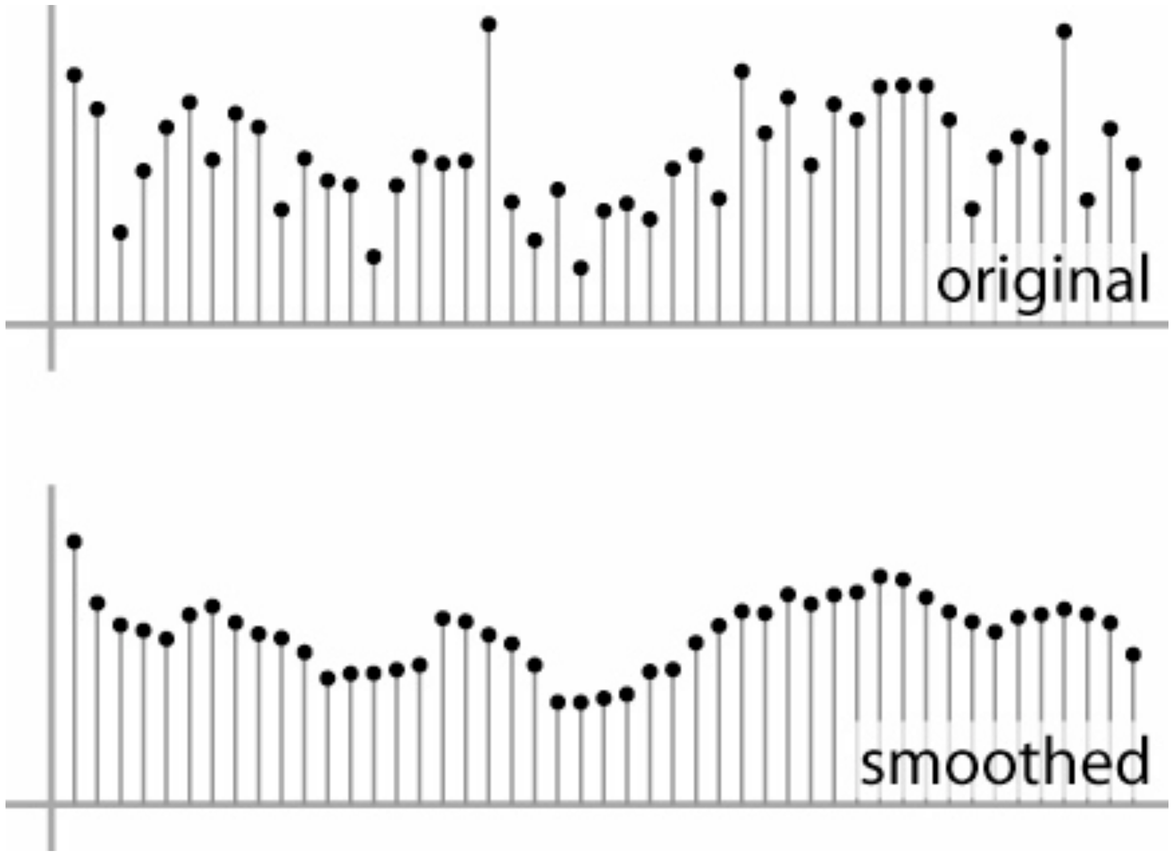


Convolution warm-up

- Same moving average operation, expressed mathematically:

$$b_{\mathrm{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

Discrete convolution

- Simple averaging:

$$b_{\text{smooth}}[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} b[j]$$

every sample gets the same weight

- Convolution: same idea but with *weighted* average

$$(a \star b)[i] = \sum_j a[j]b[i-j]$$

each sample gets its own weight (normally zero far away)

- This is all convolution is: it is a **moving weighted average**

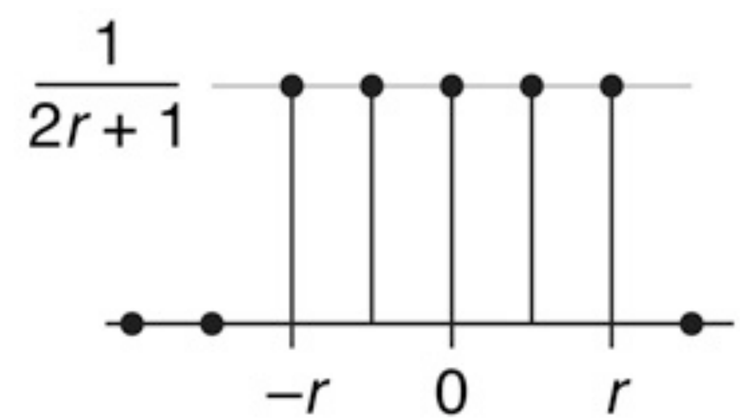
Filters

- Sequence of weights *a*[*j*] is called a *filter*
- Filter is nonzero over its *region of support*

  usually centered on zero: support radius *r*
- Filter is *normalized* so that it sums to 1.0

  this makes for a weighted average, not just any old weighted sum
- Most filters are symmetric about 0

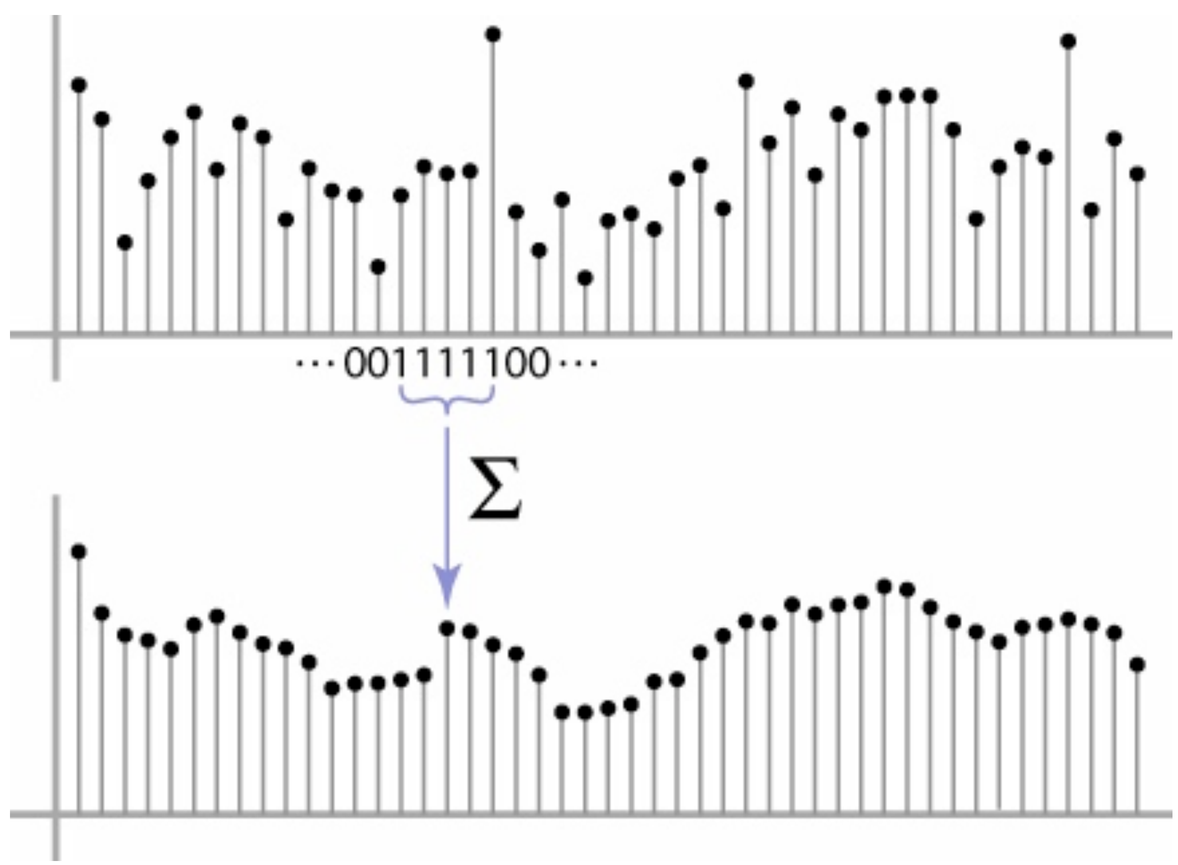
  since for images we usually want to treat left and right the same



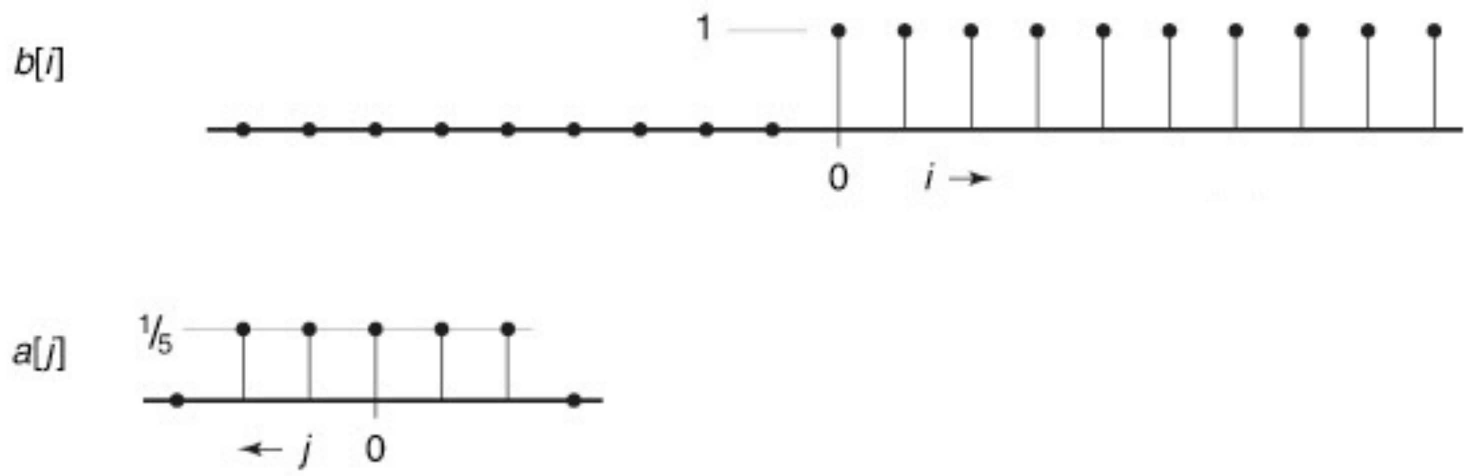
a box filter

Convolution and filtering

- Can express sliding average as convolution with a *box filter*
- $a_{\text{box}} = [\ldots, 0, 1, 1, 1, 1, 1, 0, \ldots]$



Example: box and step



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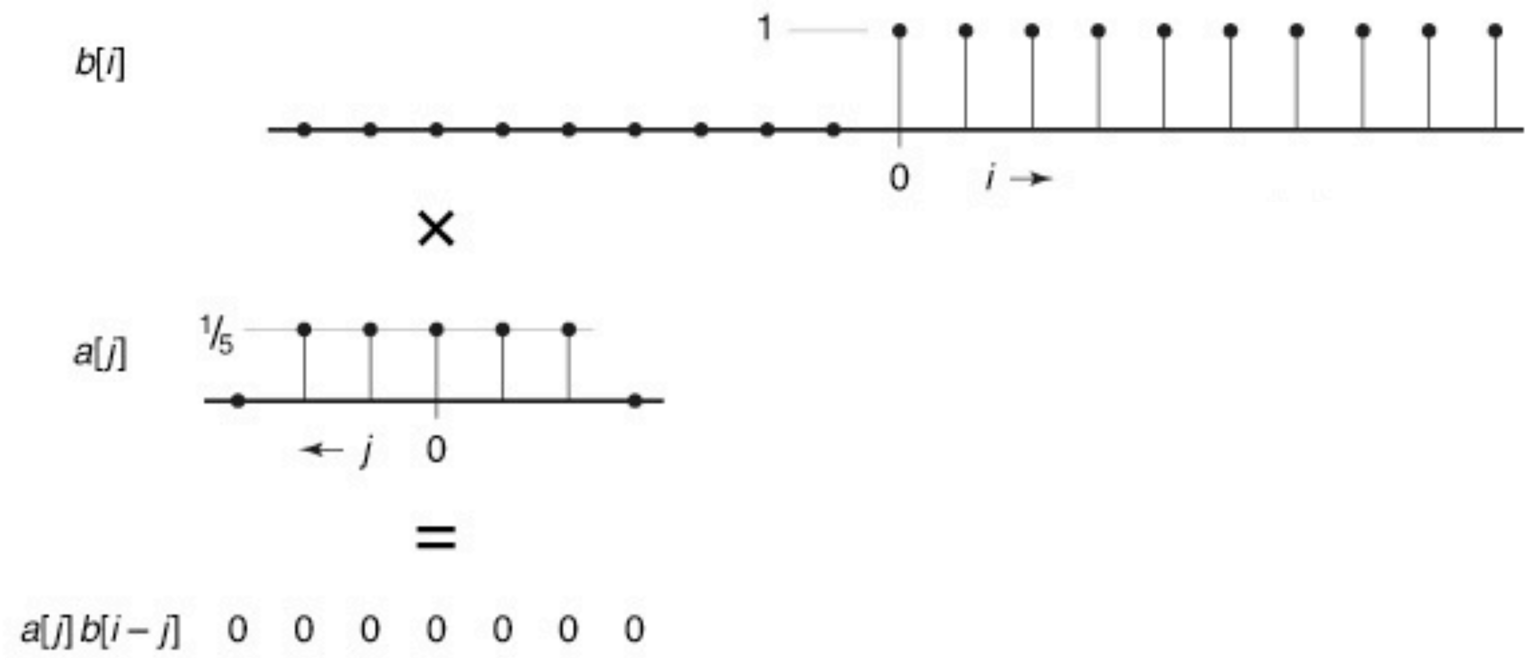

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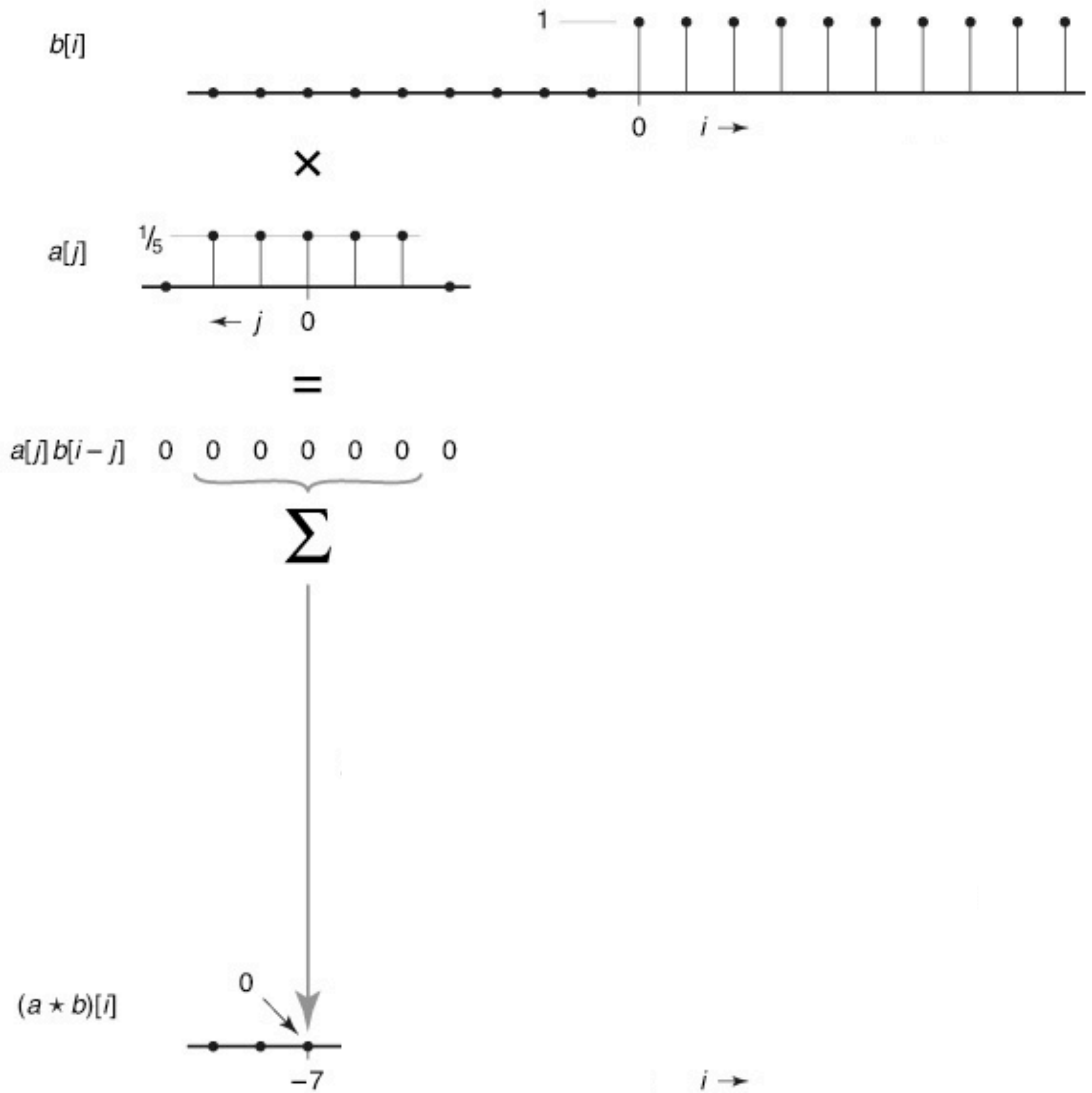

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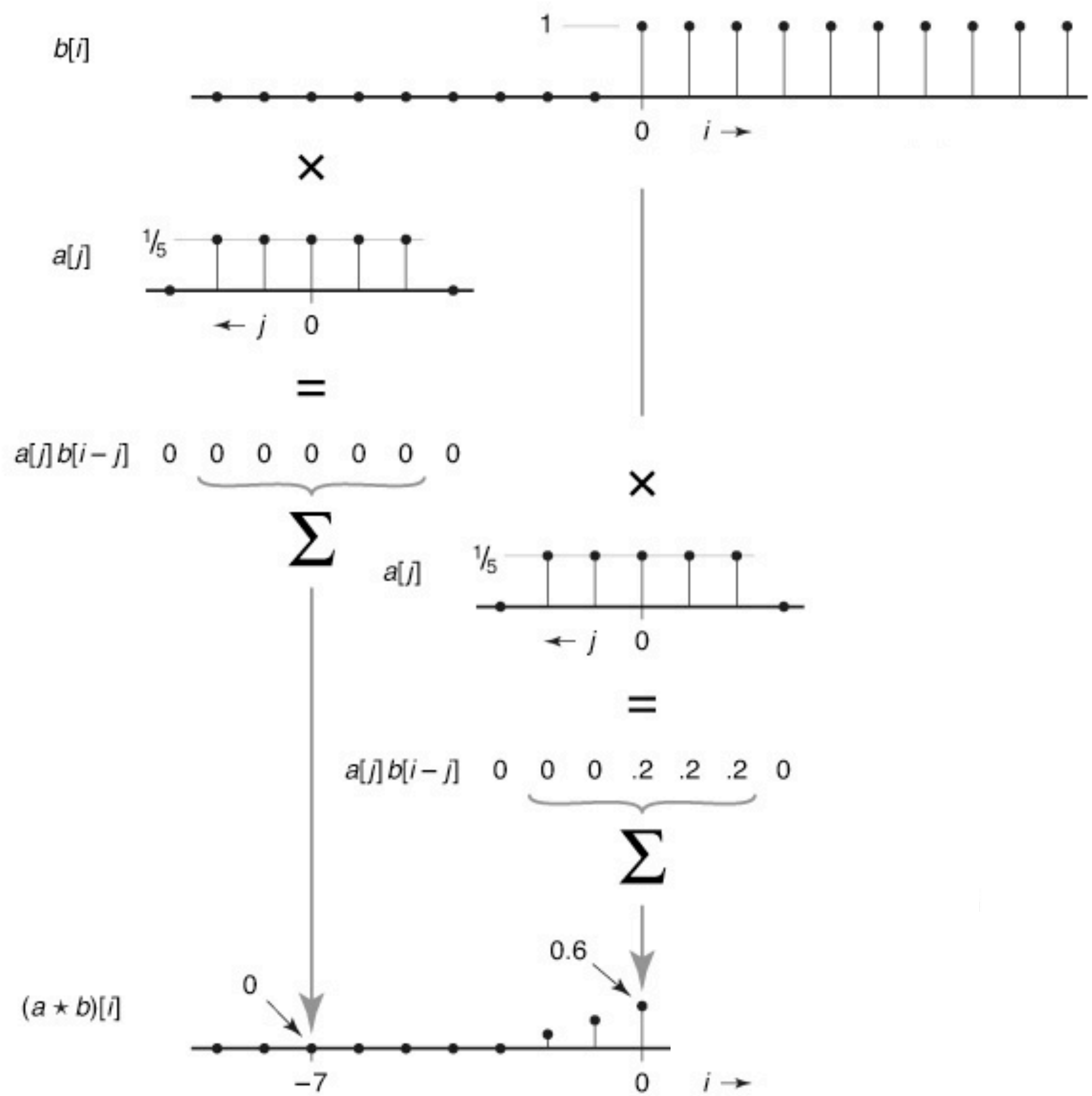

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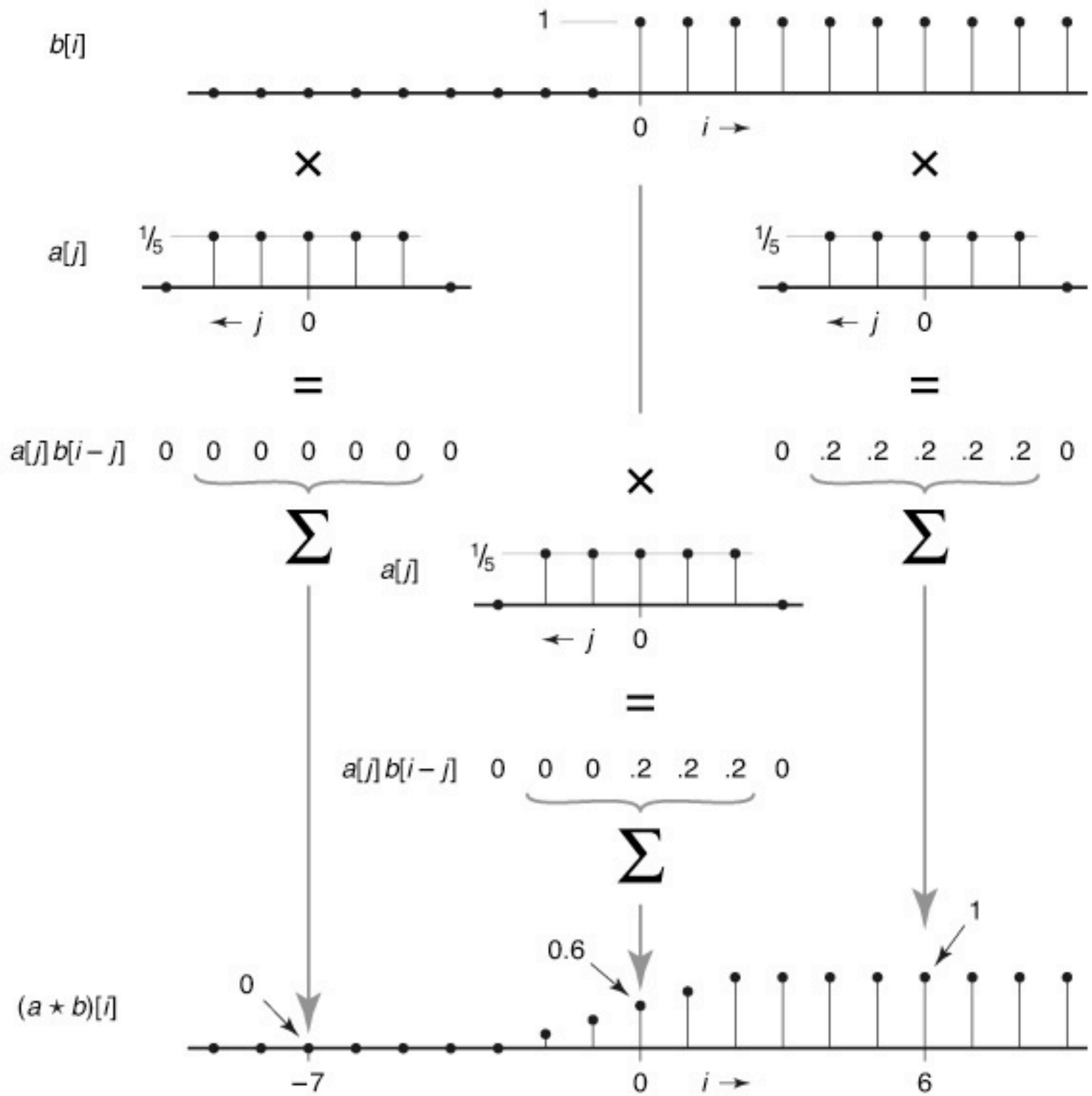
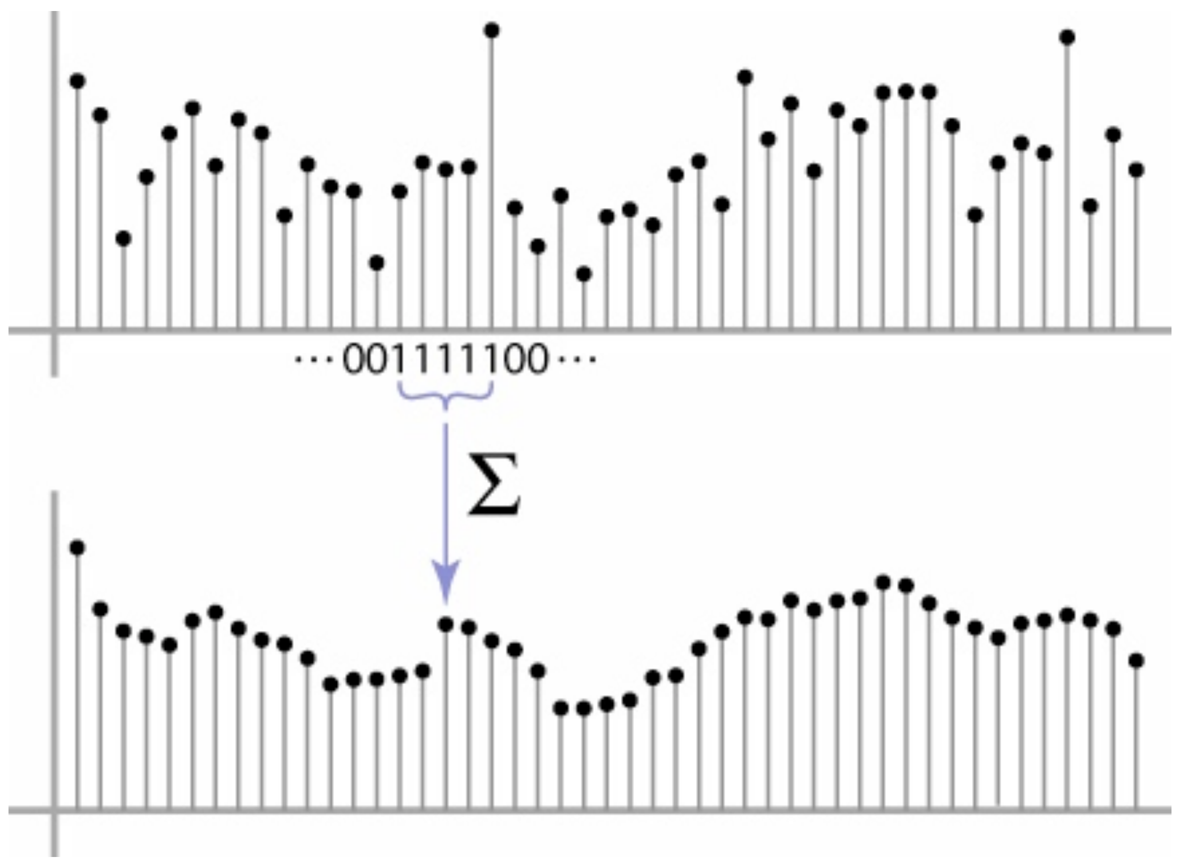


Convolution and filtering

- Convolution applies with any sequence of weights
- Example: bell curve (gaussian-like) [..., 1, 4, 6, 4, 1, ...]/16



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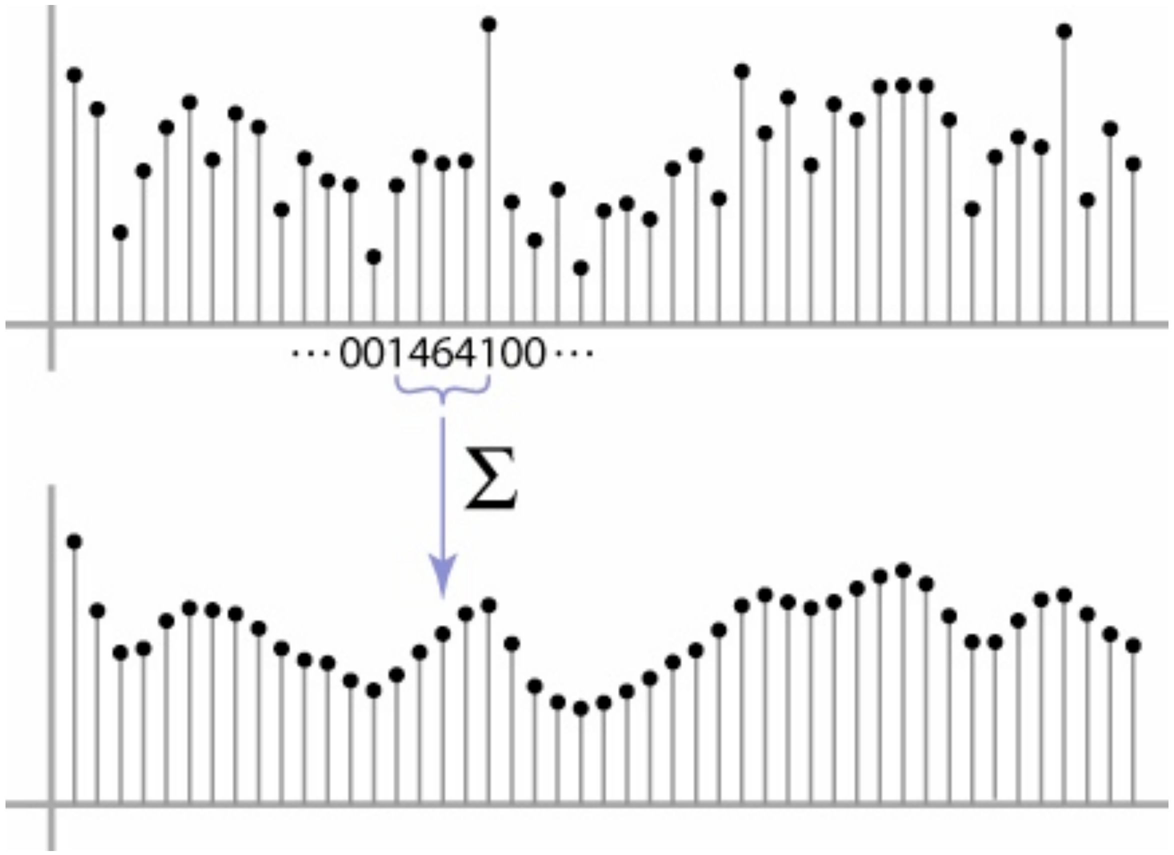

Discrete convolution

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation
  - commutative $a \star b = b \star a$
  - associative $a \star (b \star c) = (a \star b) \star c$
  - distributes over addition $a \star (b + c) = a \star b + a \star c$
  - scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$
  - identity: unit impulse e = [. . ., 0, 0, 1, 0, 0, . . .]

$$a \star e = a$$

- Conceptually no distinction between filter and signal

Discrete filtering in 2D

- Same equation, one more index

$$(a \star b)[i,j] = \sum_{i',j'} a[i',j']b[i-i',j-j']$$

  now the filter is a rectangle you slide around over a grid of numbers

- Commonly applied to images

  blurring (using box, using gaussian, …)

  sharpening (impulse minus blur)

- Usefulness of associativity

  often apply several filters one after another: $(((a * b_1) * b_2) * b_3)$

  this is equivalent to applying one filter: $a * (b_1 * b_2 * b_3)$

[Philip Greenspun]

original ▲ | ▼ box blur

sharpened ▲ | ▼ gaussian blur

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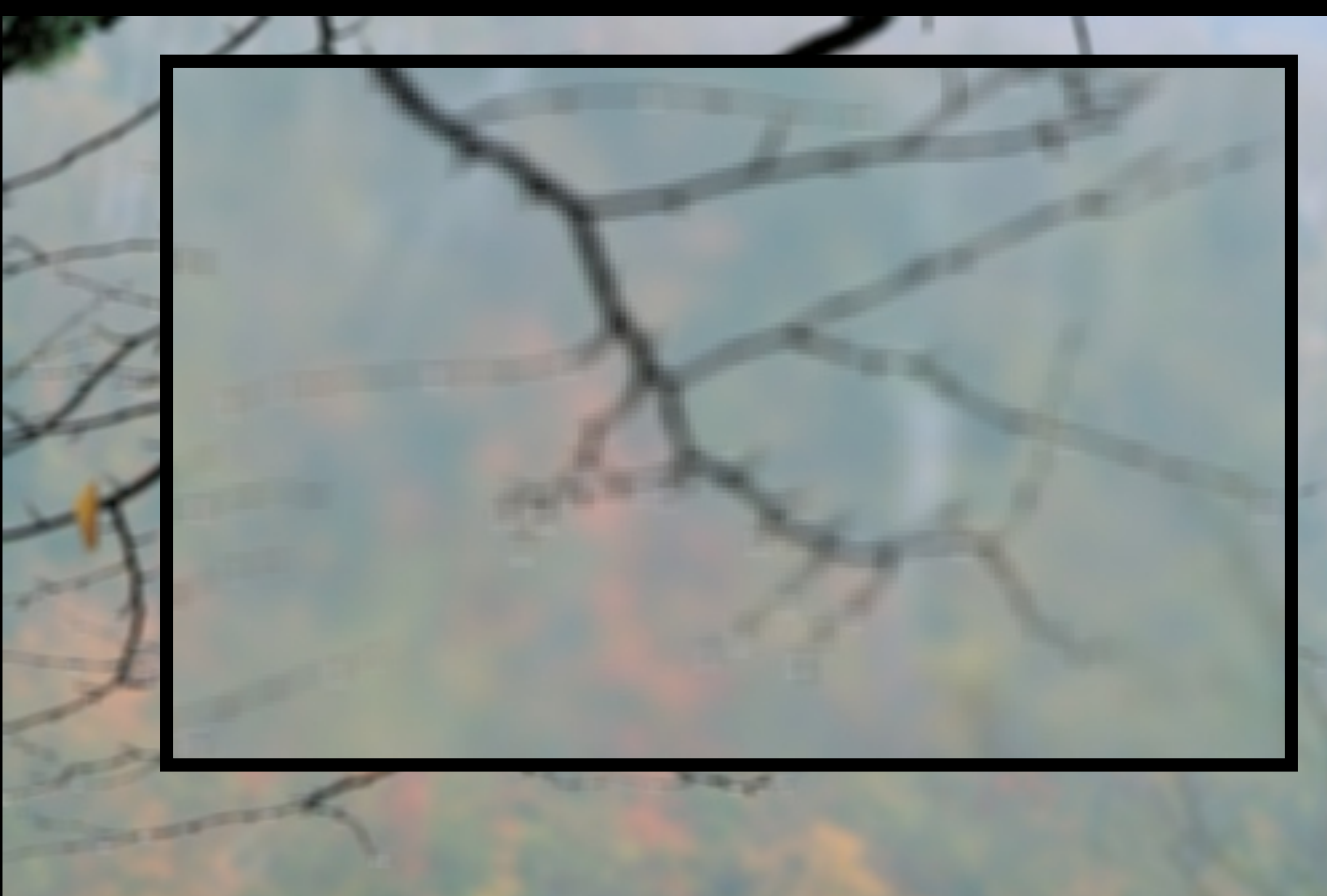
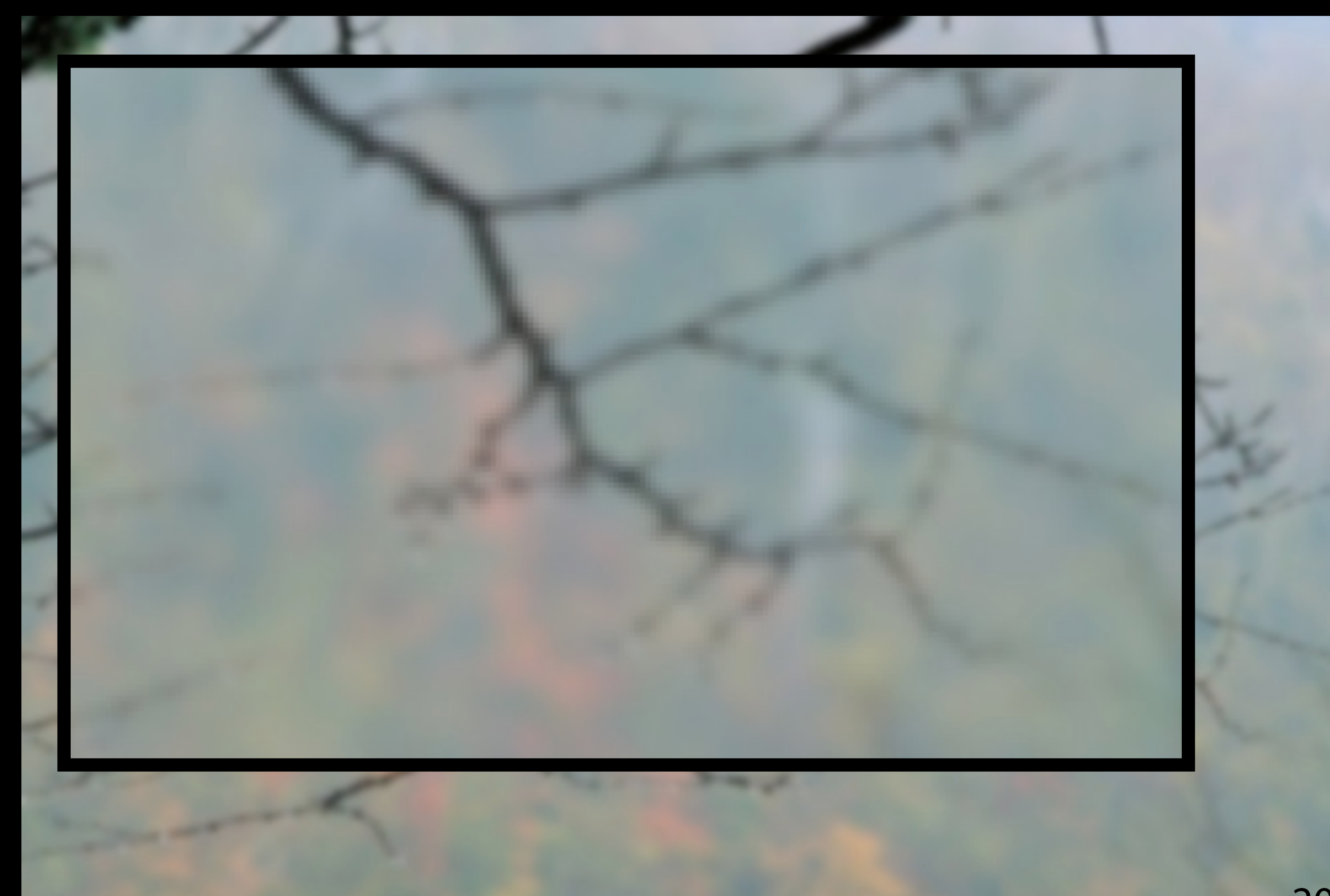
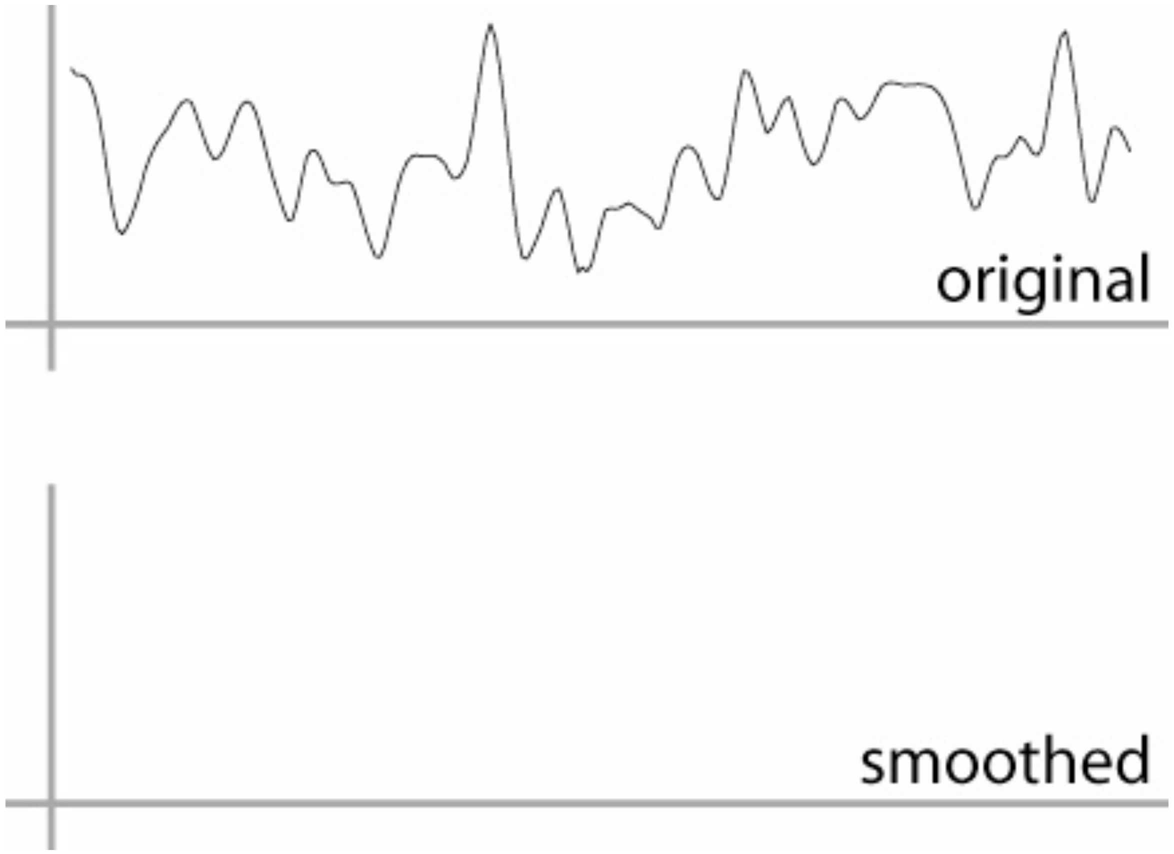




Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well
  - output is continuous
  - integration replaces summation



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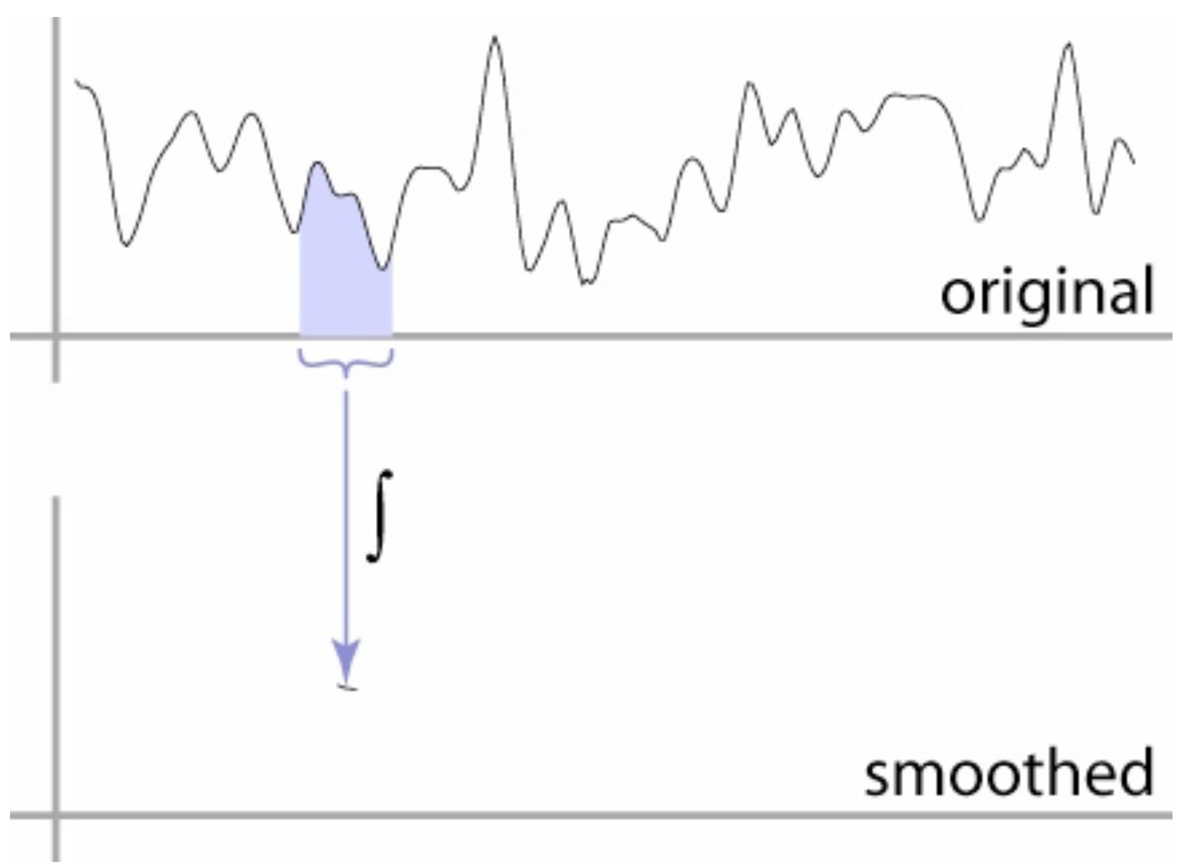

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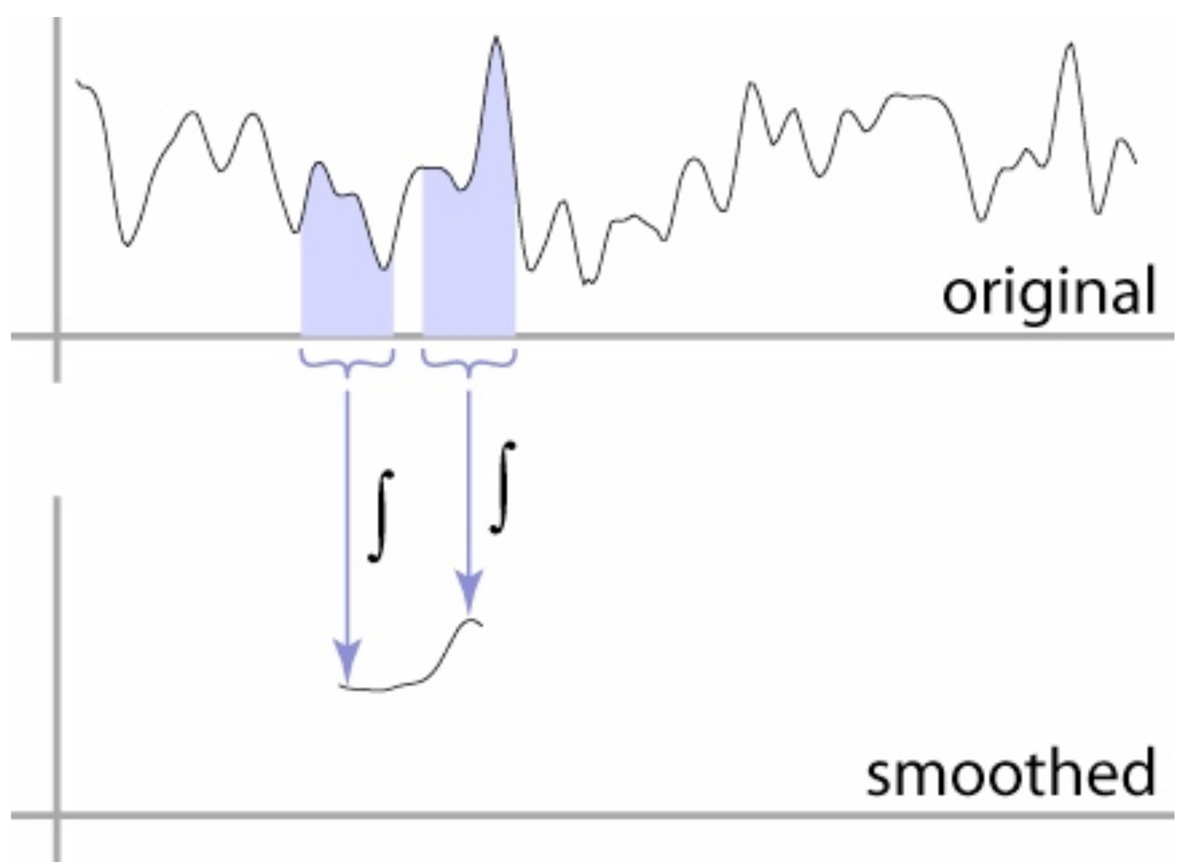

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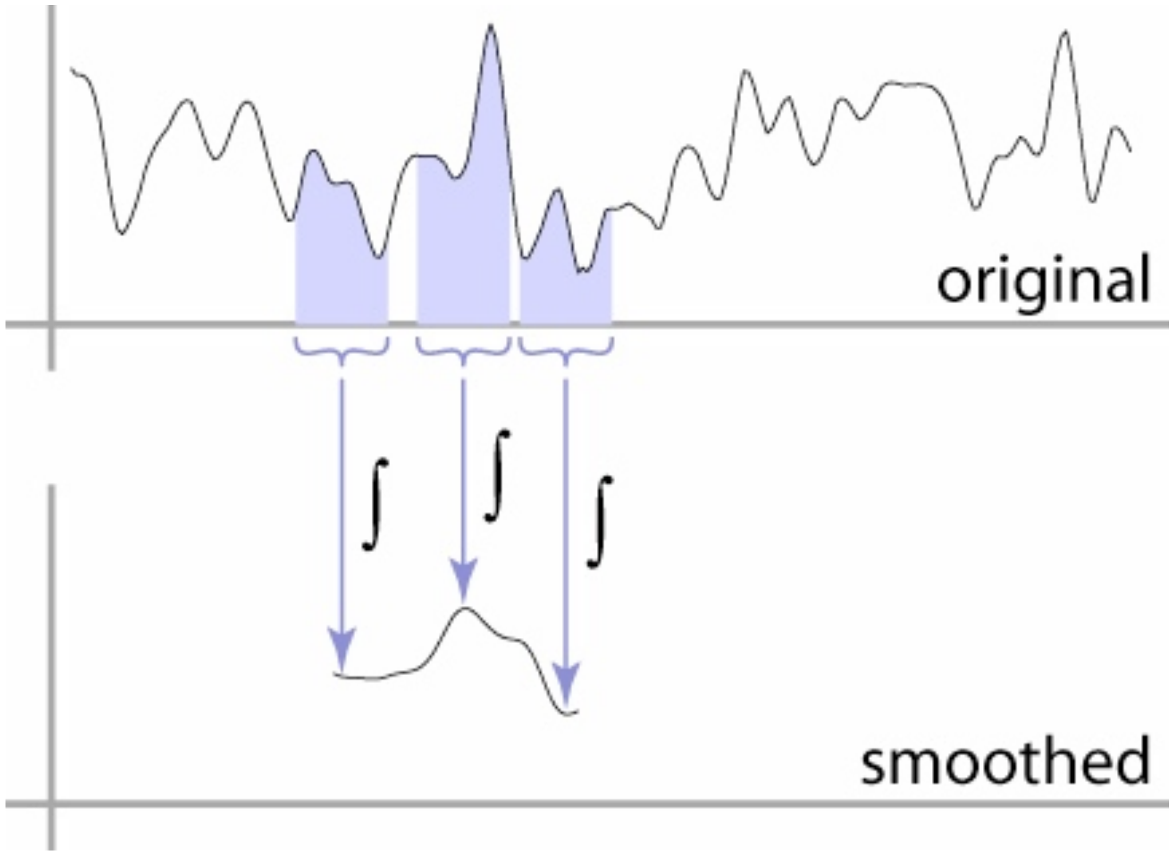

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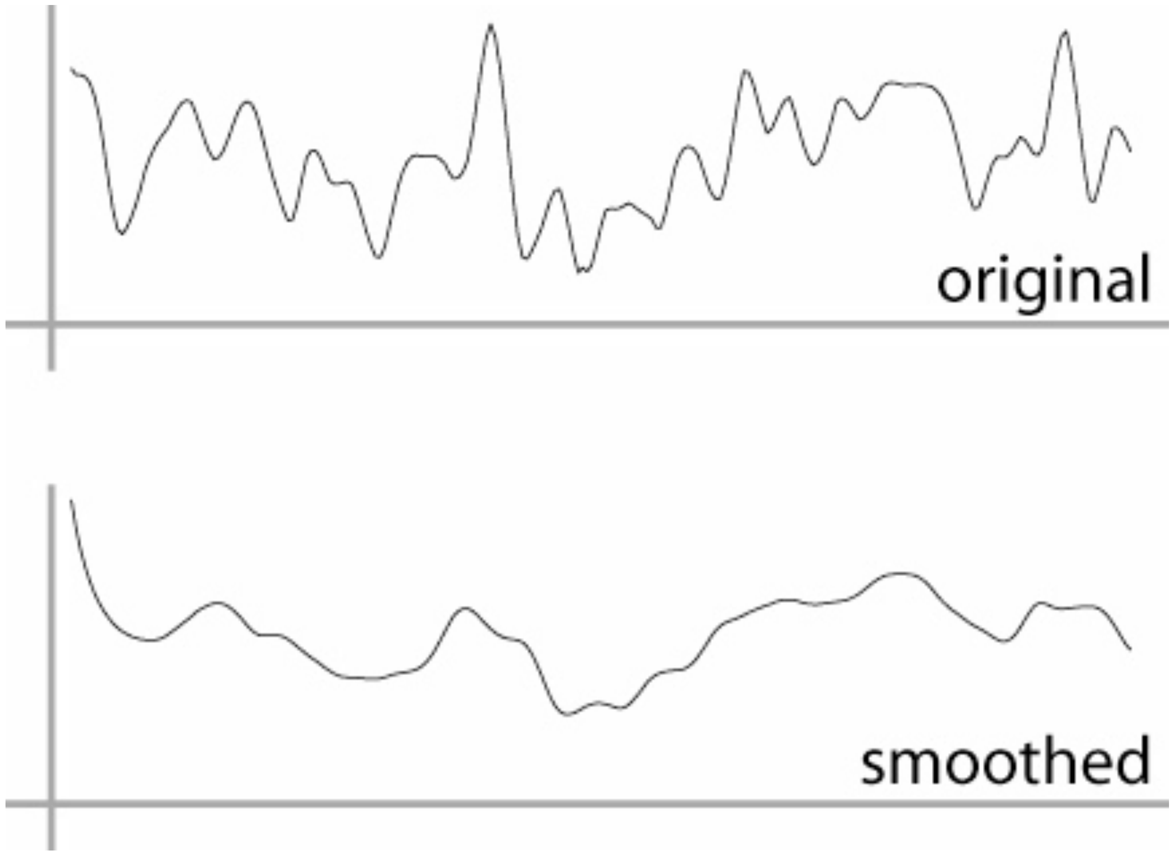


Continuous convolution

- Sliding average expressed mathematically:

$$g_{\mathrm{smooth}}(x) = \frac{1}{2r}\int_{x-r}^{x+r} g(t)dt$$

  note difference in normalization (only for box)

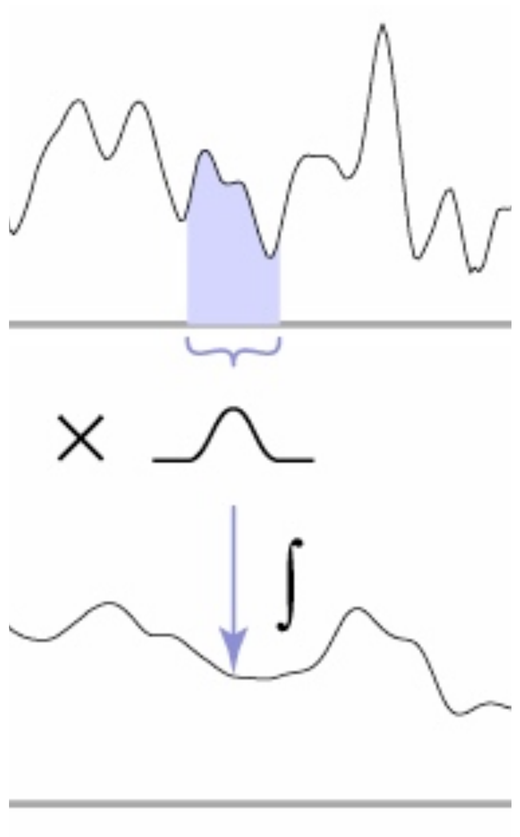
- Convolution just adds weights

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

  weighting is now by a function

  weighted integral is like weighted average

  again bounds are set by support of *f(x)*



One more convolution

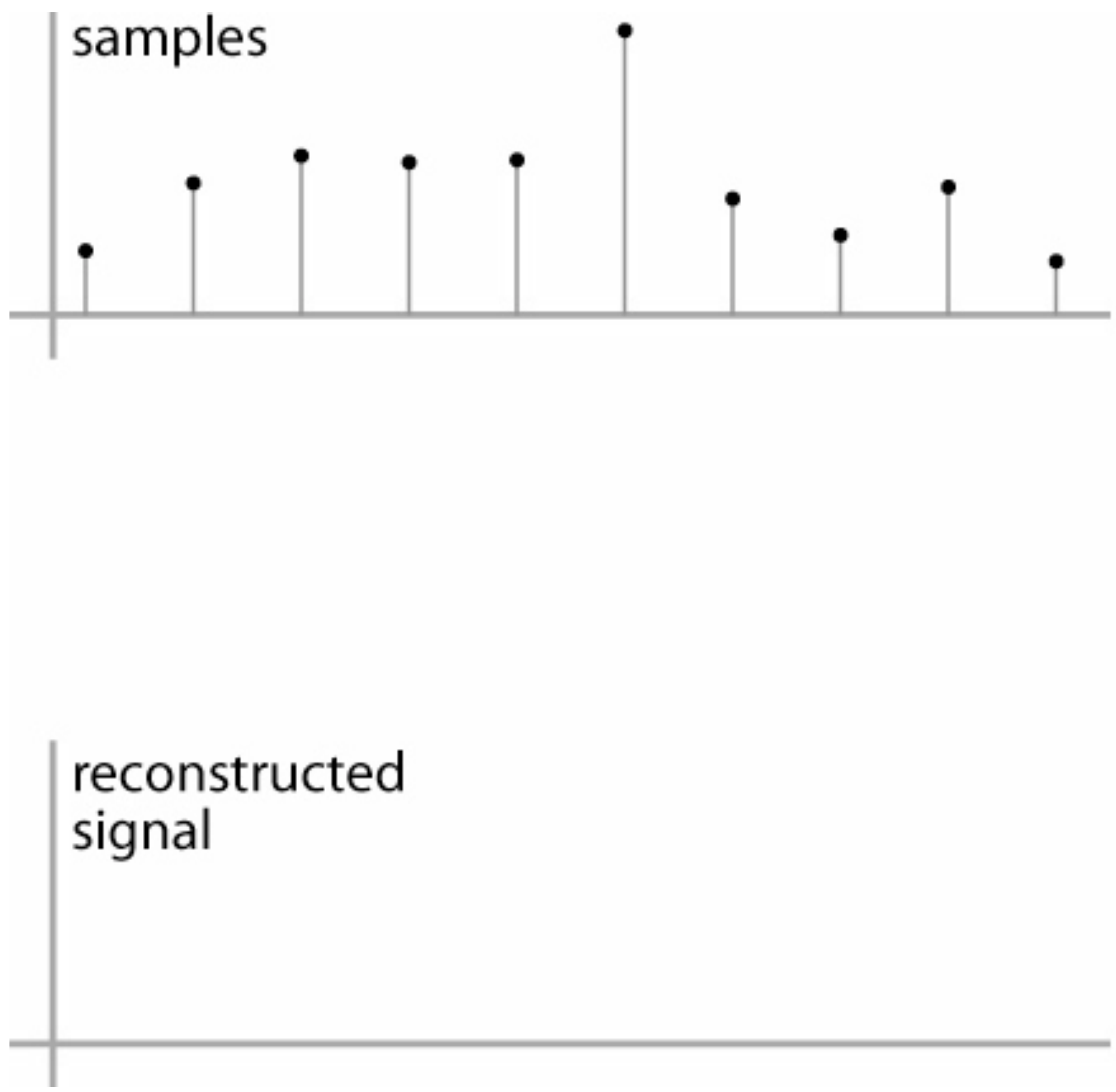
- Continuous–discrete convolution

$$(a \star f)(x) = \sum_i a[i] f(x - i)$$

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$

used for reconstruction and resampling

Continuous-discrete convolution



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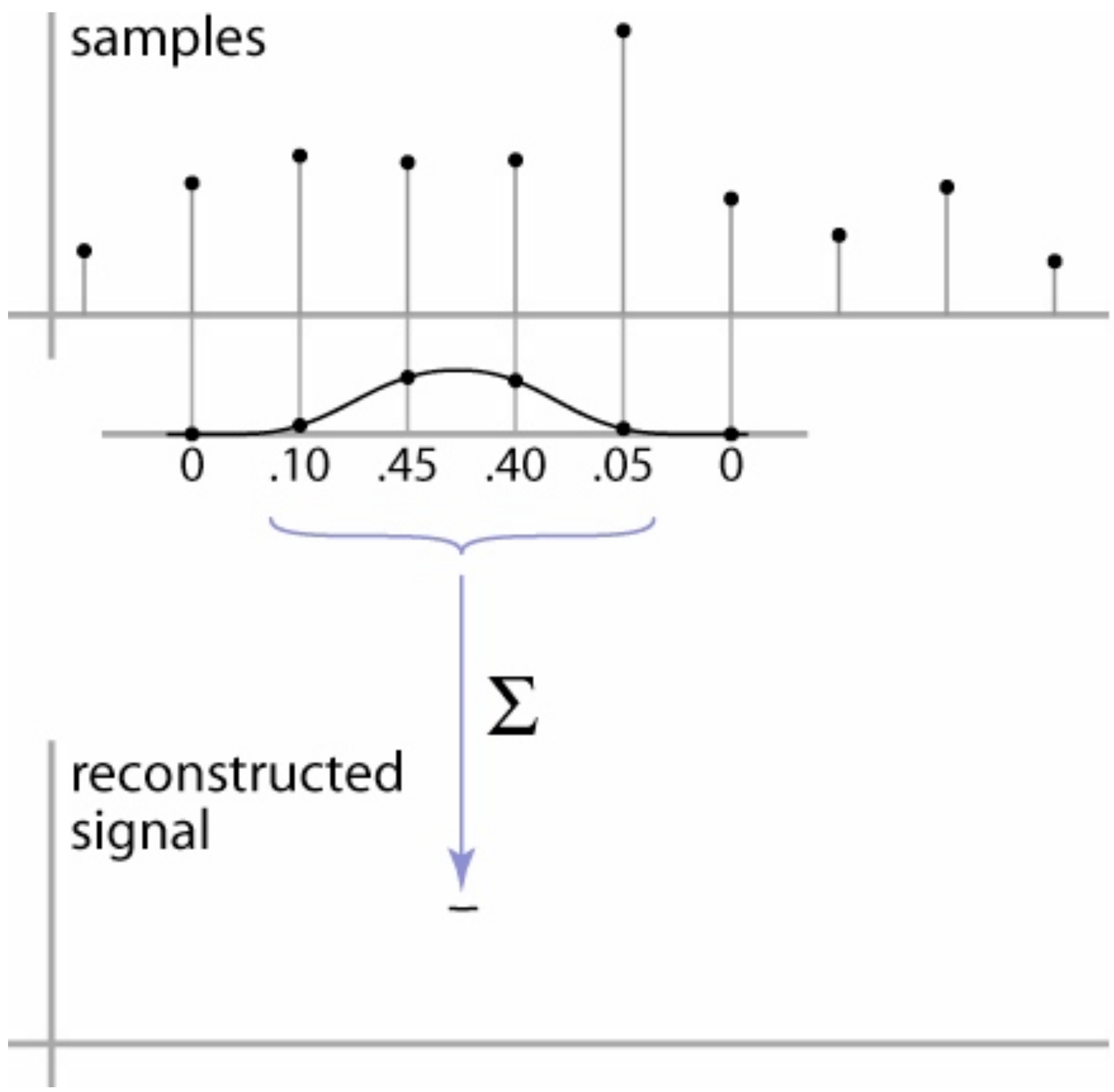

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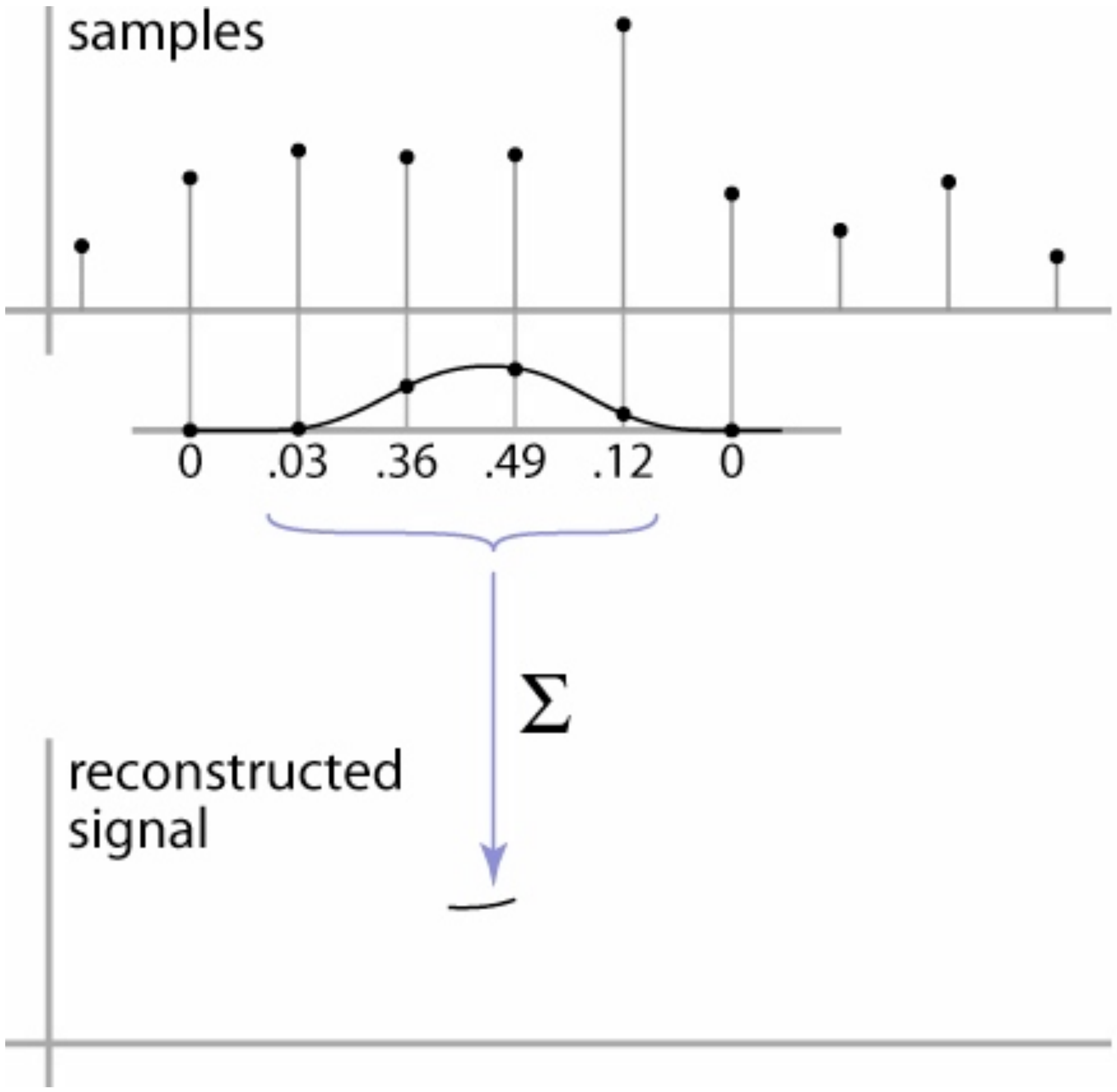

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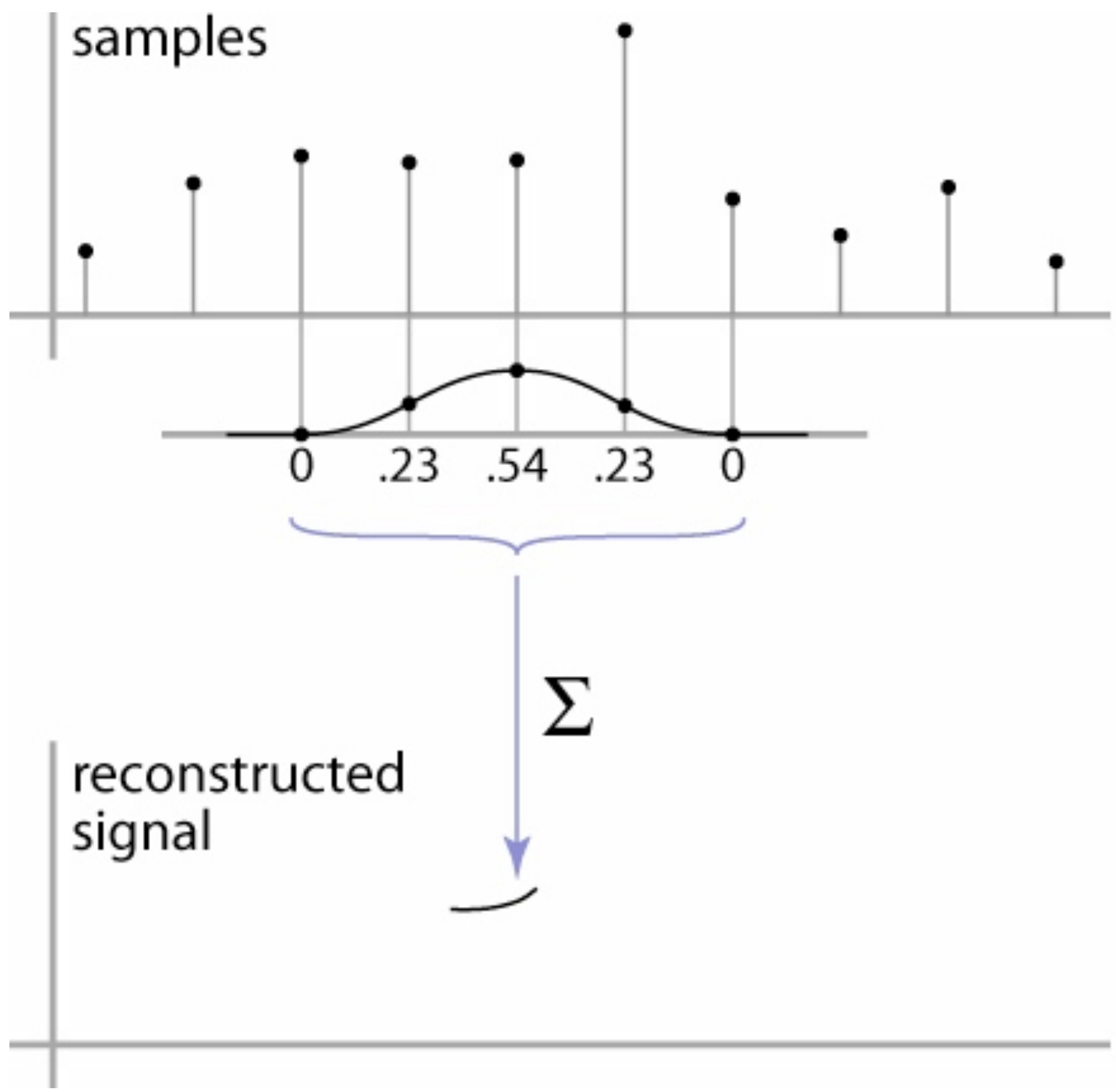

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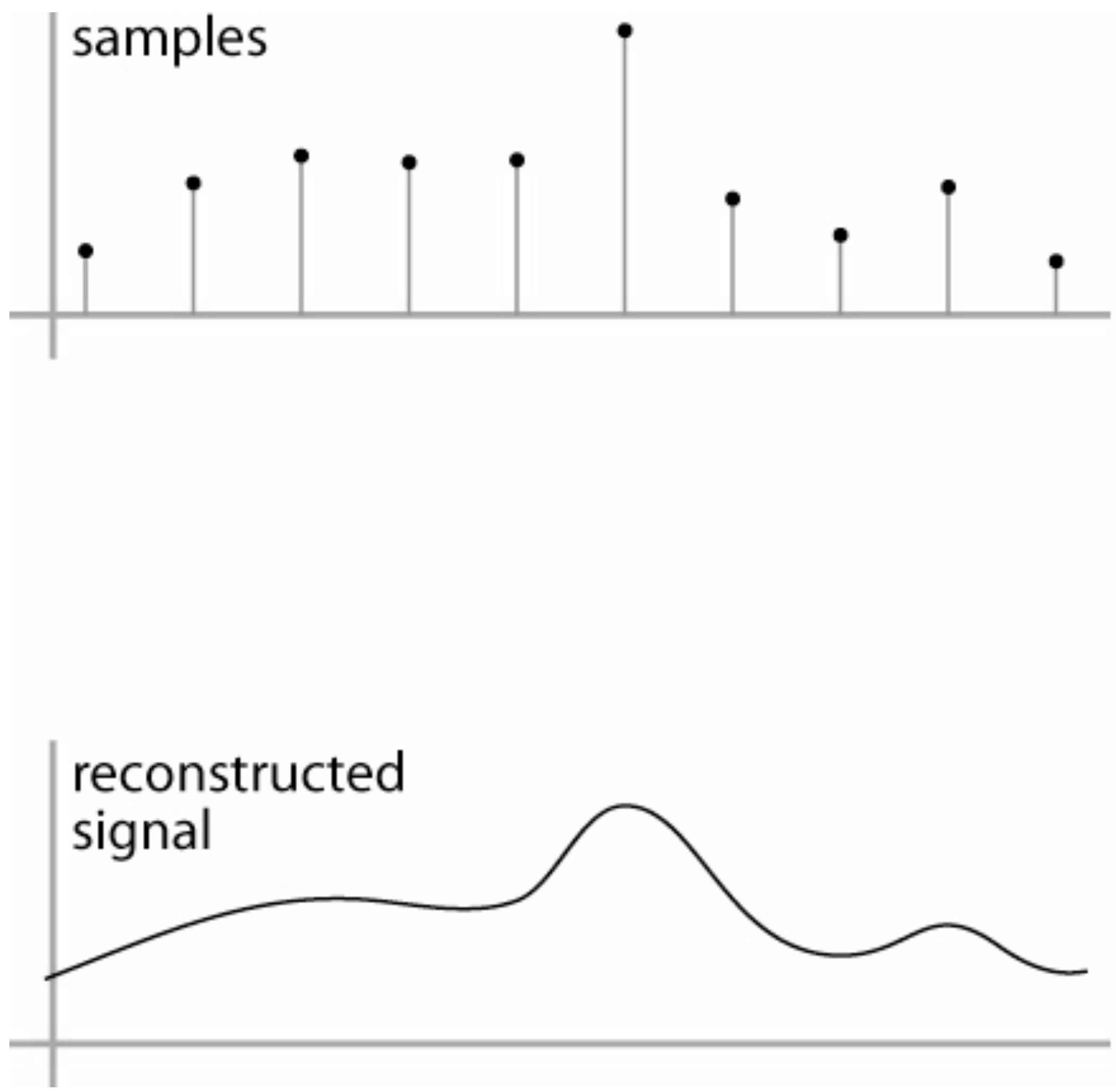
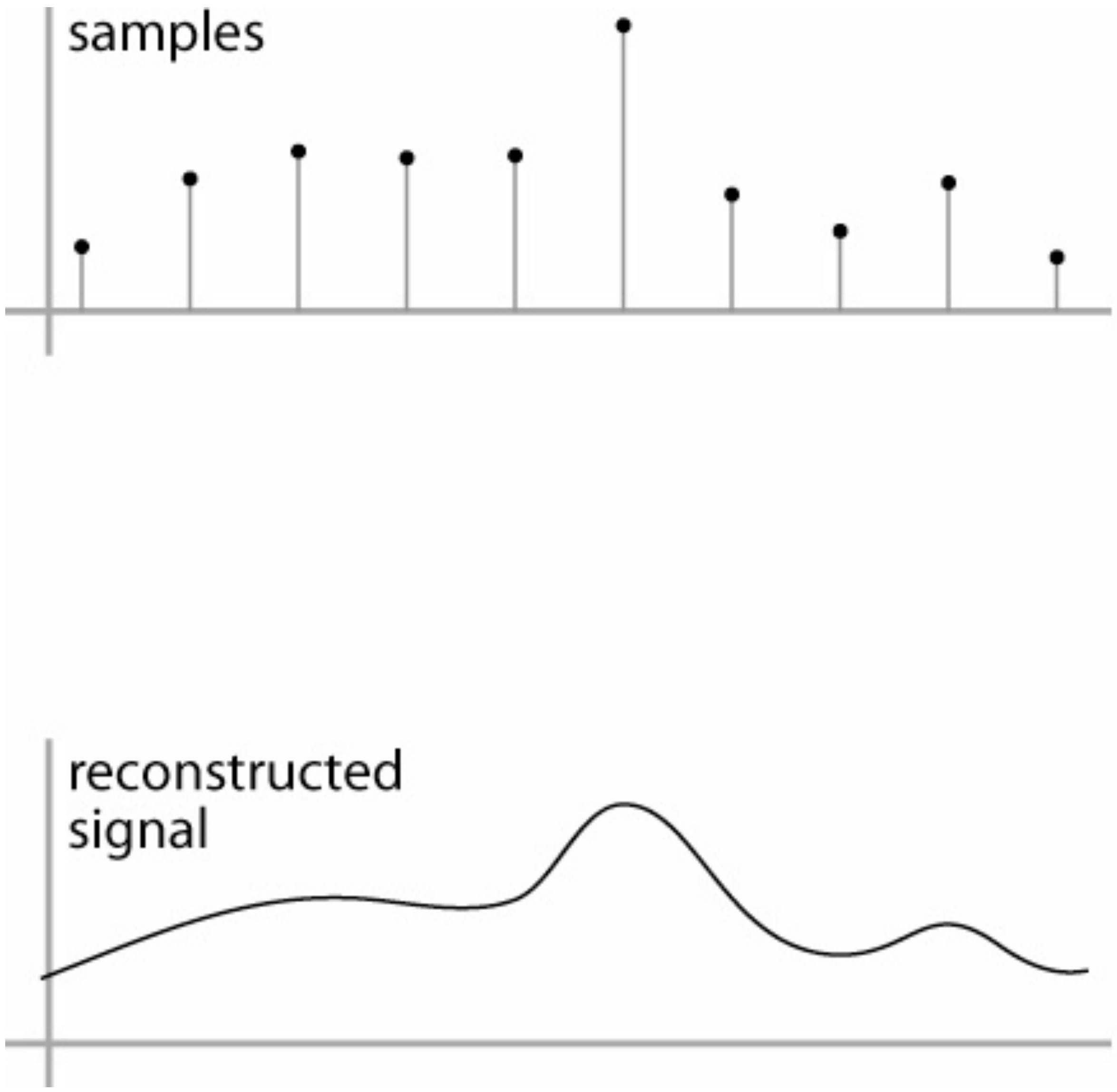


Resampling

- Changing the sample rate
  - in images, this is enlarging and reducing
- Creating more samples:
  - increasing the sample rate
  - “upsampling”
  - “enlarging”
- Ending up with fewer samples:
  - decreasing the sample rate
  - “downsampling”
  - “reducing”

Resampling

- Reconstruction creates a continuous function
  - forget its origins, go ahead and sample it



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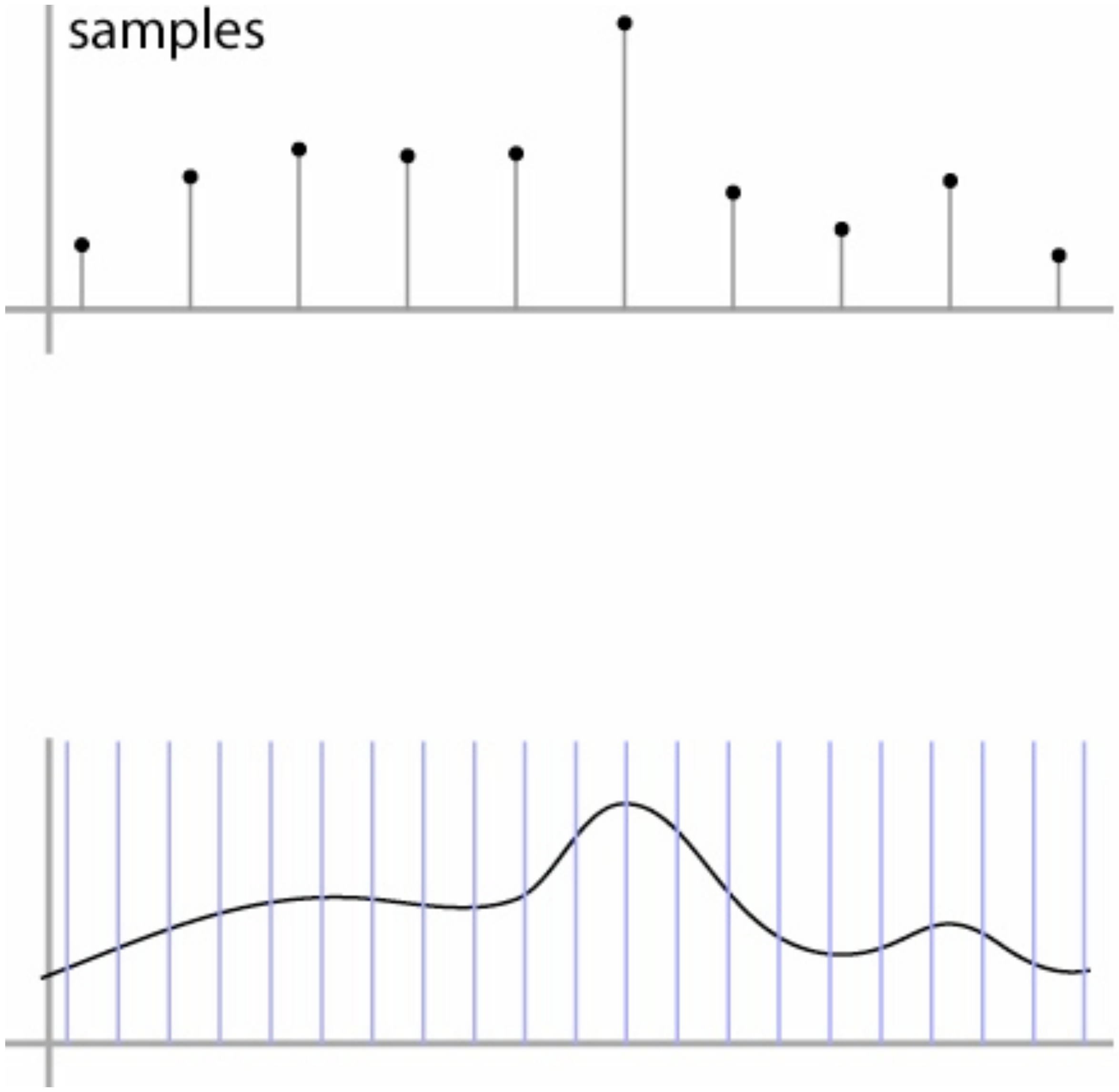

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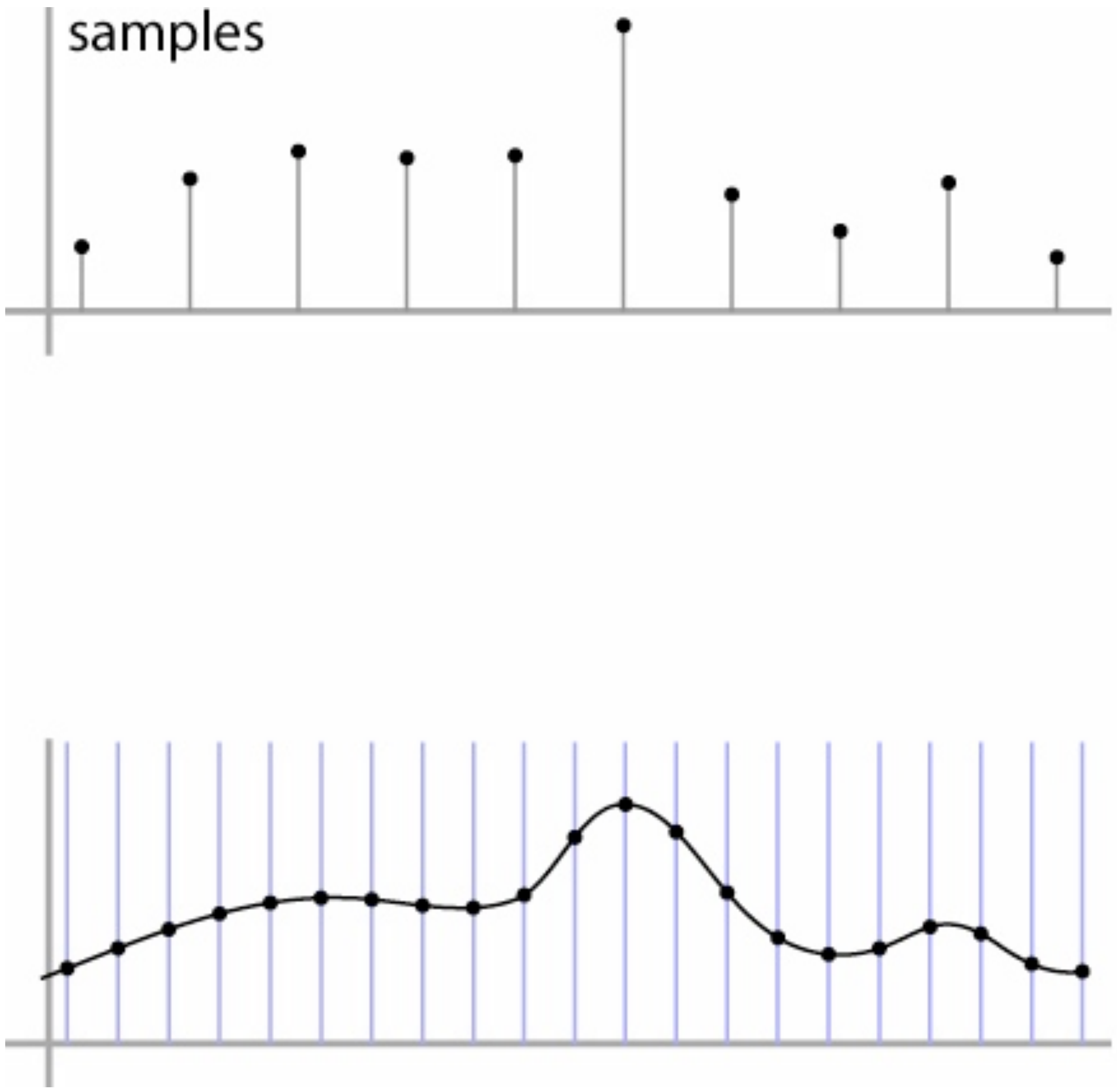

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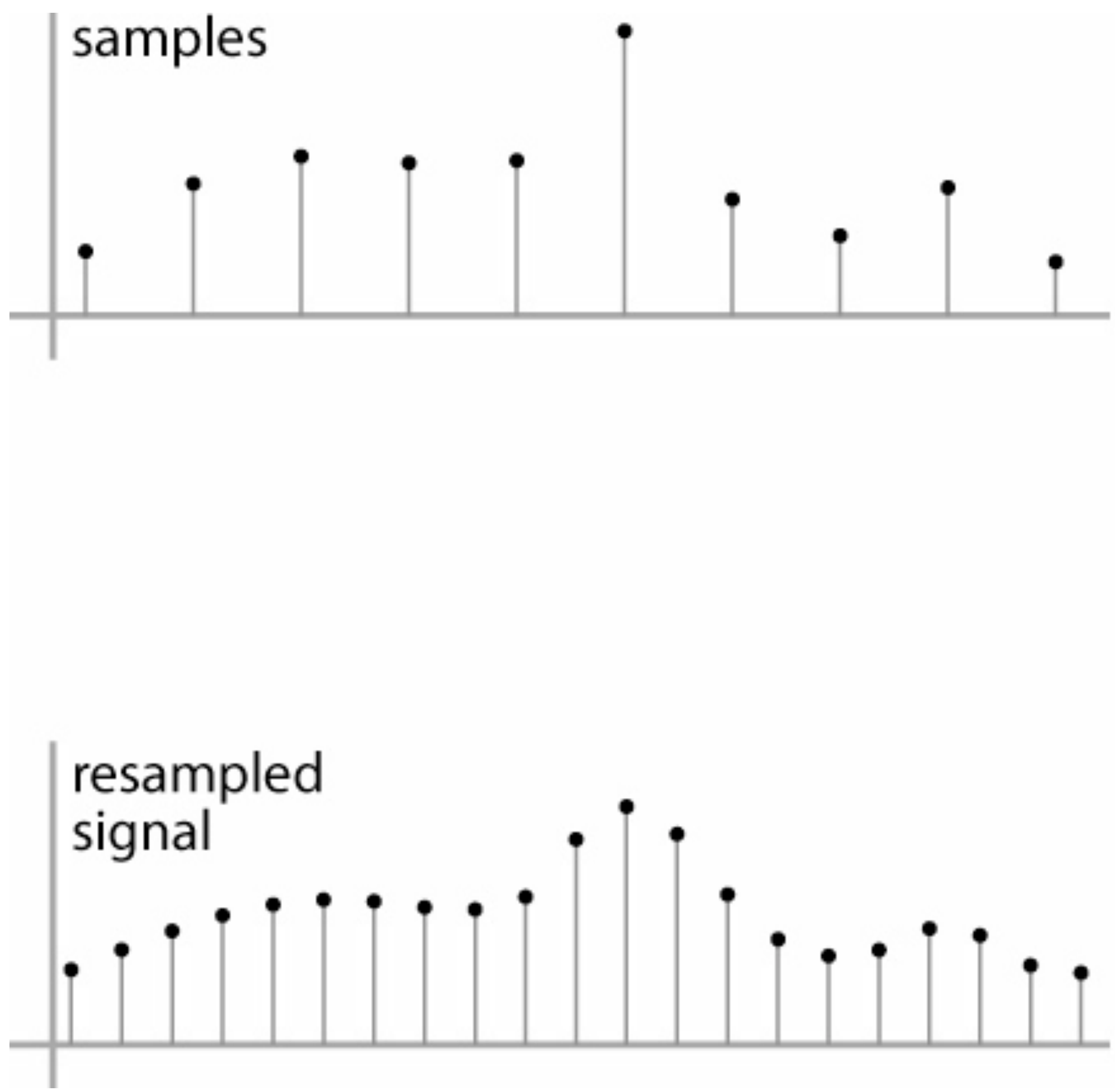

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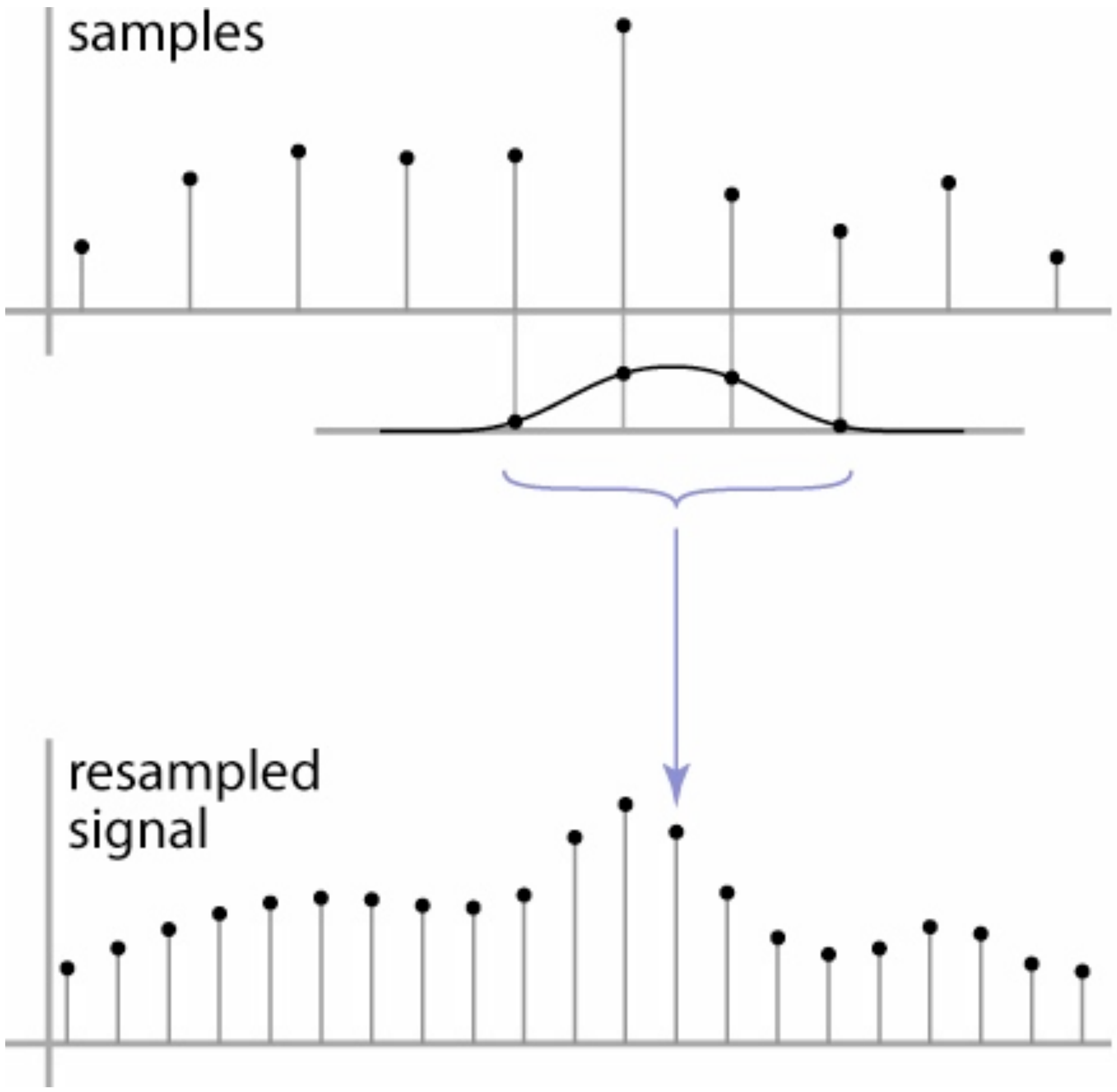


Cont.–disc. convolution in 2D

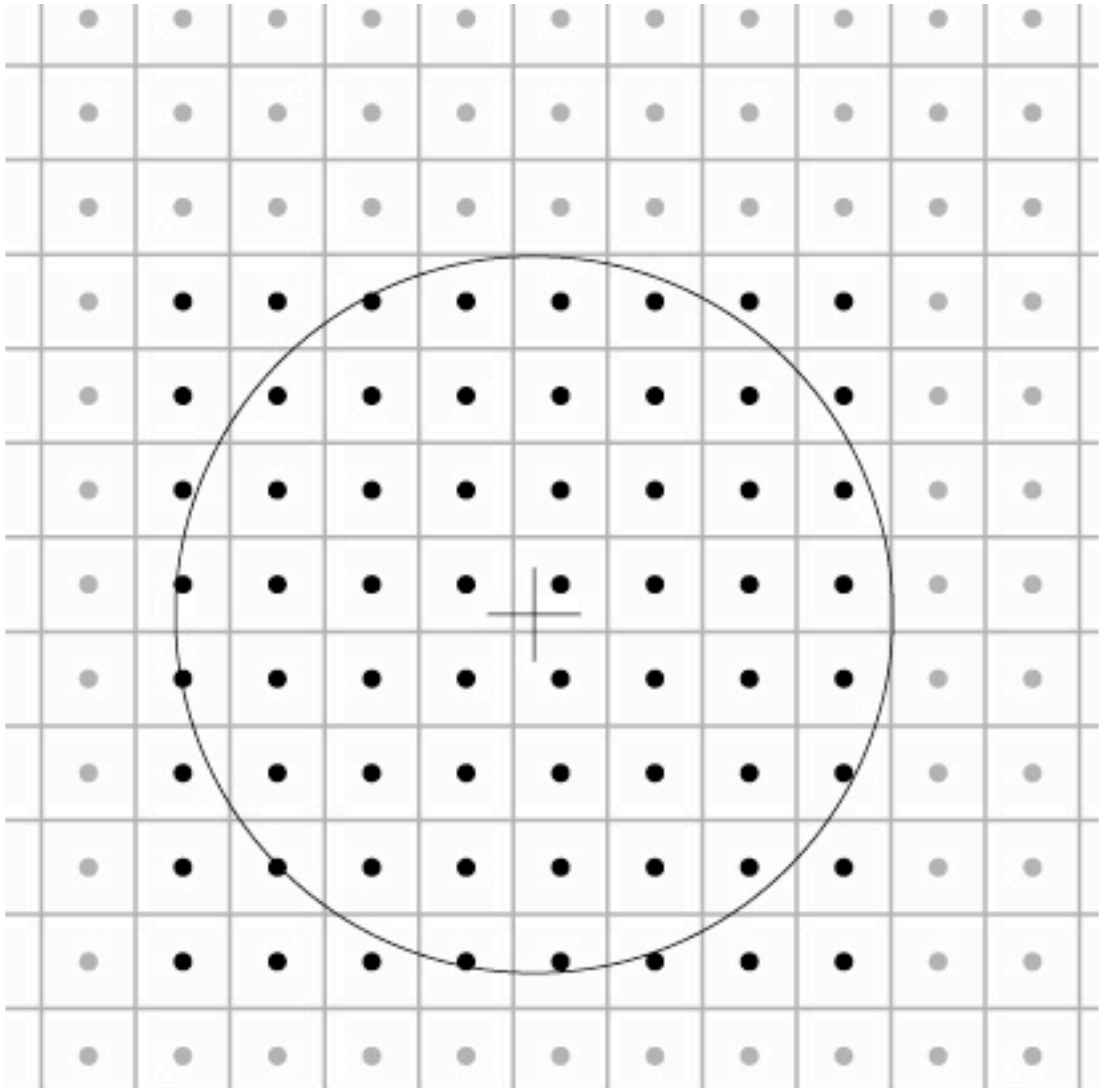
- same convolution—just two variables now

$$(a \star f)(x, y) = \sum_{i,j} a[i, j] f(x - i, y - j)$$

loop over nearby pixels,
average using filter weight

looks like discrete filter,
but offsets are not integers
and filter is continuous

remember placement of filter
relative to grid is variable



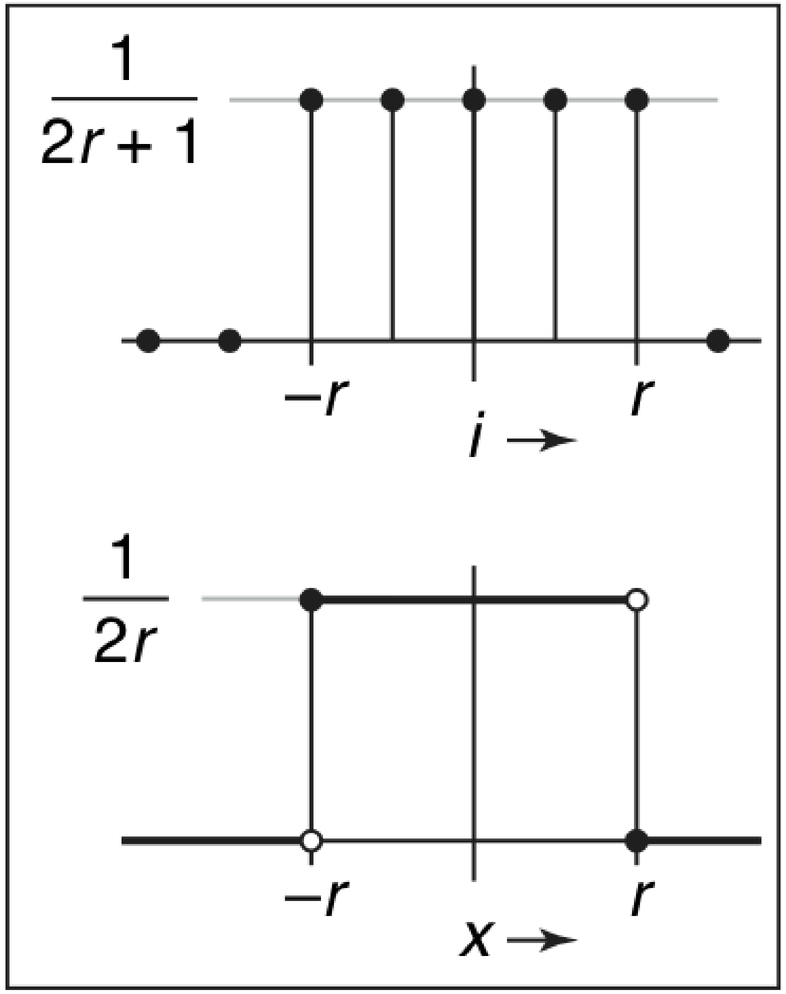
A gallery of filters

- Box filter
  Simple and cheap
- Tent filter
  Linear interpolation
- Gaussian filter
  Very smooth antialiasing filter
- B-spline cubic
  Very smooth
- Catmull-rom cubic
  Interpolating
- Mitchell-Netravali cubic
  Good for image upsampling

Box filter

$$a_{\text{box},r}[i] = \begin{cases} 1/(2r+1) & |i| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

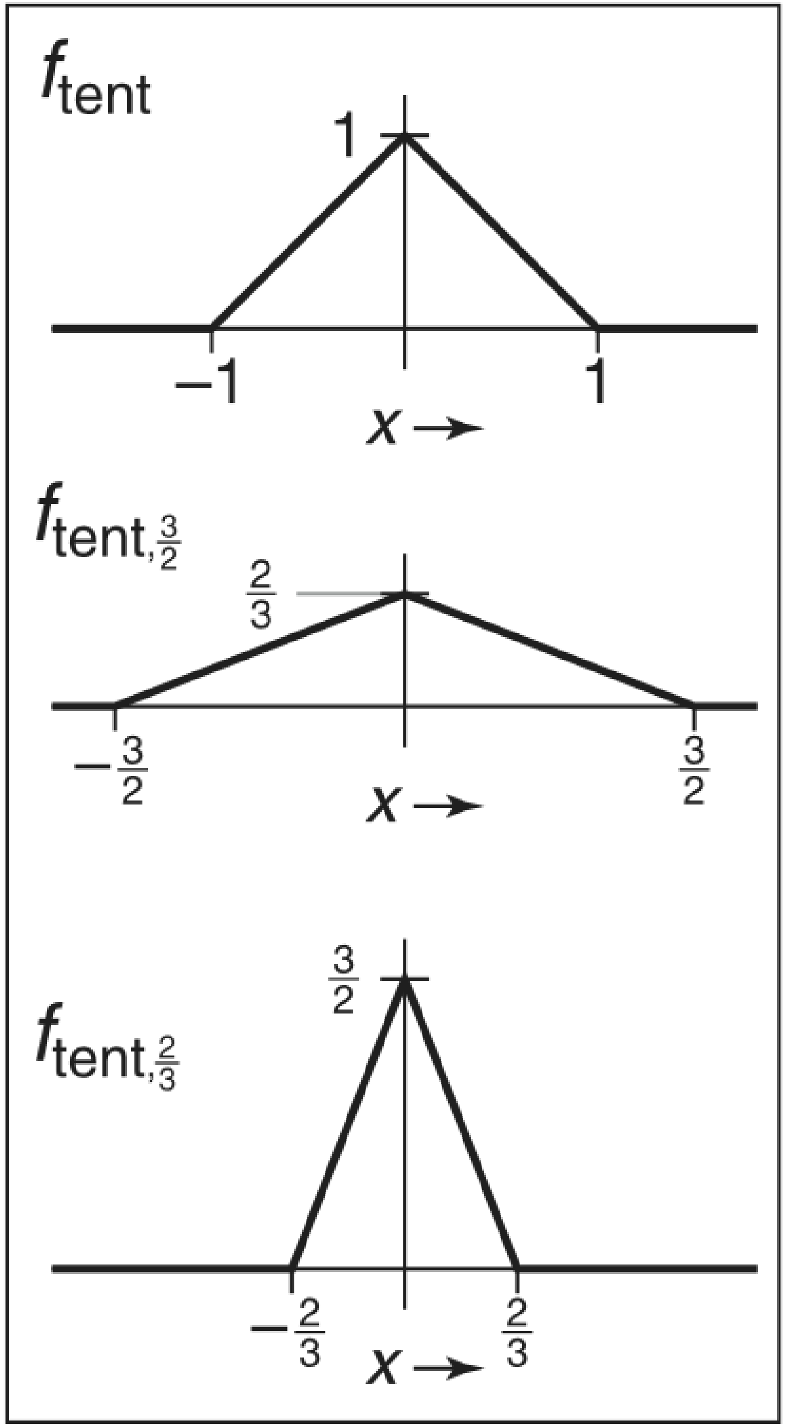
$$f_{\text{box},r}(x) = \begin{cases} 1/(2r) & -r \le x < r, \\ 0 & \text{otherwise.} \end{cases}$$



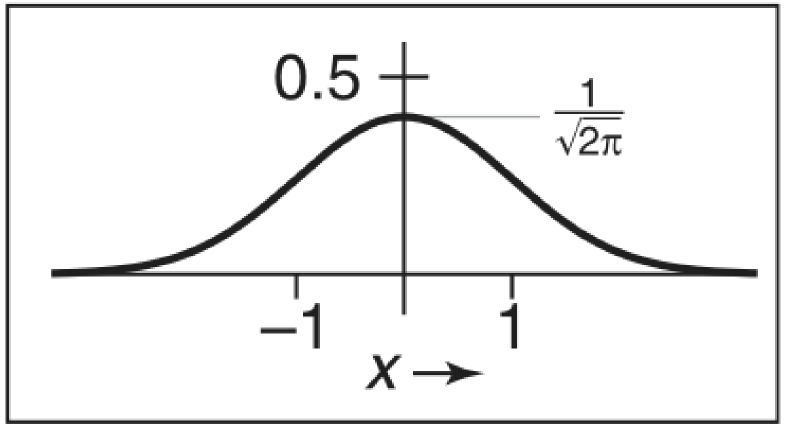
Tent filter

$$f_{\text{tent}}(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$f_{\text{tent},r}(x) = \frac{f_{\text{tent}}(x/r)}{r}.$$

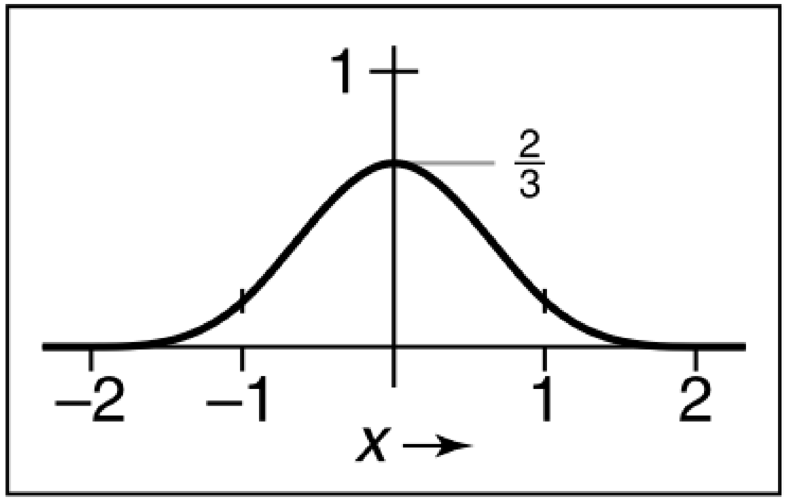


Gaussian filter



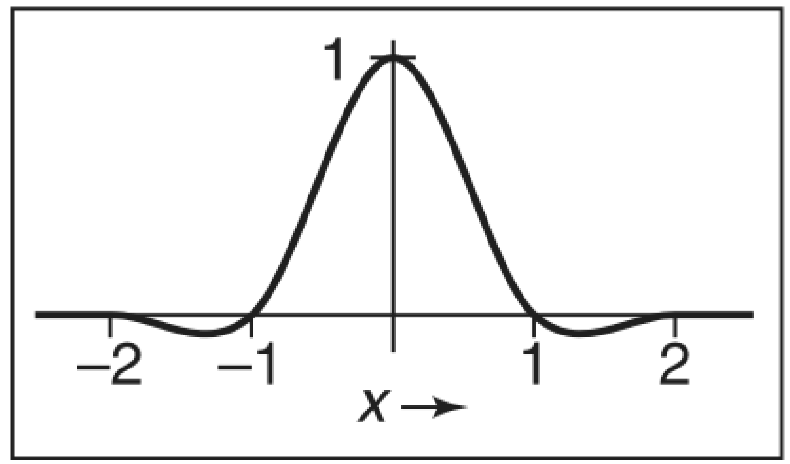
$$f_g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

B-Spline cubic



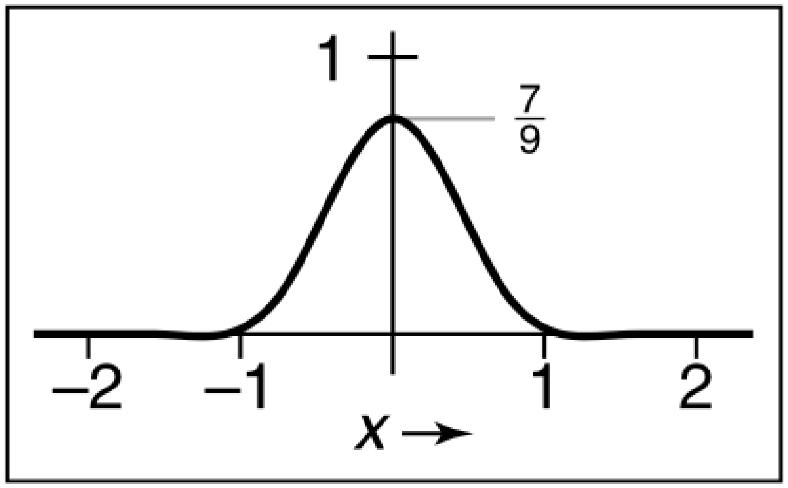
$$f_B(x) = \frac{1}{6}\begin{cases} -3(1-|x|)^3 + 3(1-|x|)^2 + 3(1-|x|) + 1 & -1 \le x \le 1, \\ (2-|x|)^3 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Catmull-Rom cubic



$$f_C(x) = \frac{1}{2} \begin{cases} -3(1-|x|)^3 + 4(1-|x|)^2 + (1-|x|) & -1 \le x \le 1, \\ (2-|x|)^3 - (2-|x|)^2 & 1 \le |x| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Michell-Netravali cubic



$$f_M(x) = \frac{1}{3} f_B(x) + \frac{2}{3} f_C(x)$$

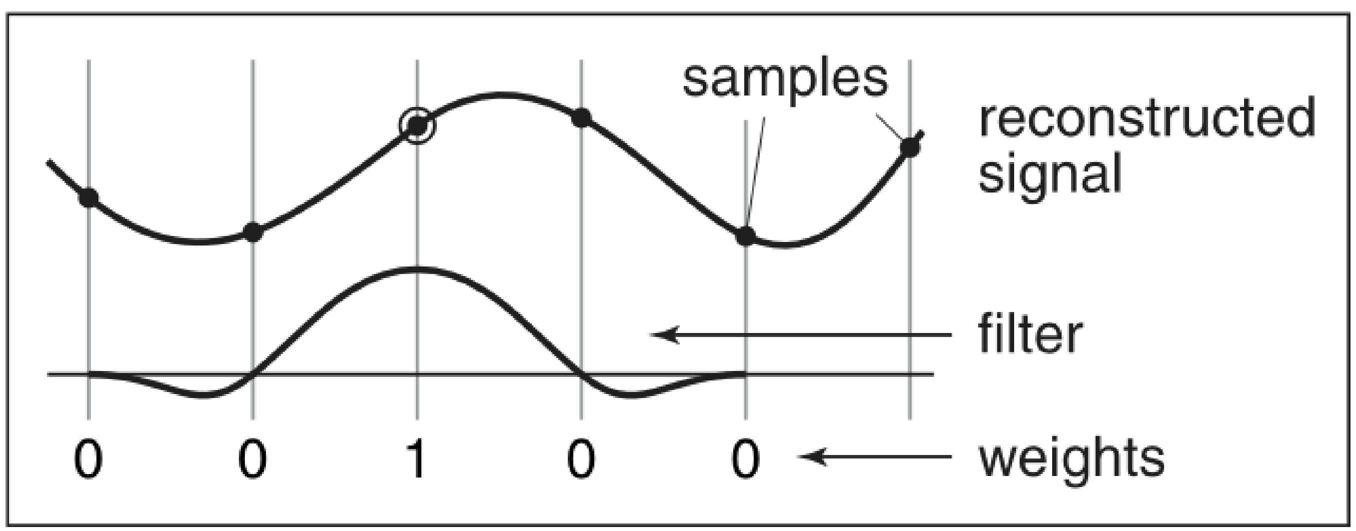
$$= \frac{1}{18} \begin{cases} -21(1-|x|)^3 + 27(1-|x|)^2 + 9(1-|x|) + 1 & -1 \leq x \leq 1, \\ 7(2-|x|)^3 - 6(2-|x|)^2 & 1 \leq |x| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Effects of reconstruction filters

- For some filters, the reconstruction process winds up implementing a simple algorithm
- Box filter (radius 0.5): nearest neighbor sampling
  - box always catches exactly one input point
  - it is the input point nearest the output point
  - so output[i, j] = input[round(x(i)), round(y(j))]
    - x(i) computes the position of the output coordinate i on the input grid
- Tent filter (radius 1): linear interpolation
  - tent catches exactly 2 input points
  - weights are a and (1 – a)
  - result is straight-line interpolation from one point to the next

Properties of filters

- Degree of continuity
- Impulse response
- Interpolating or no
- Ringing, or overshoot

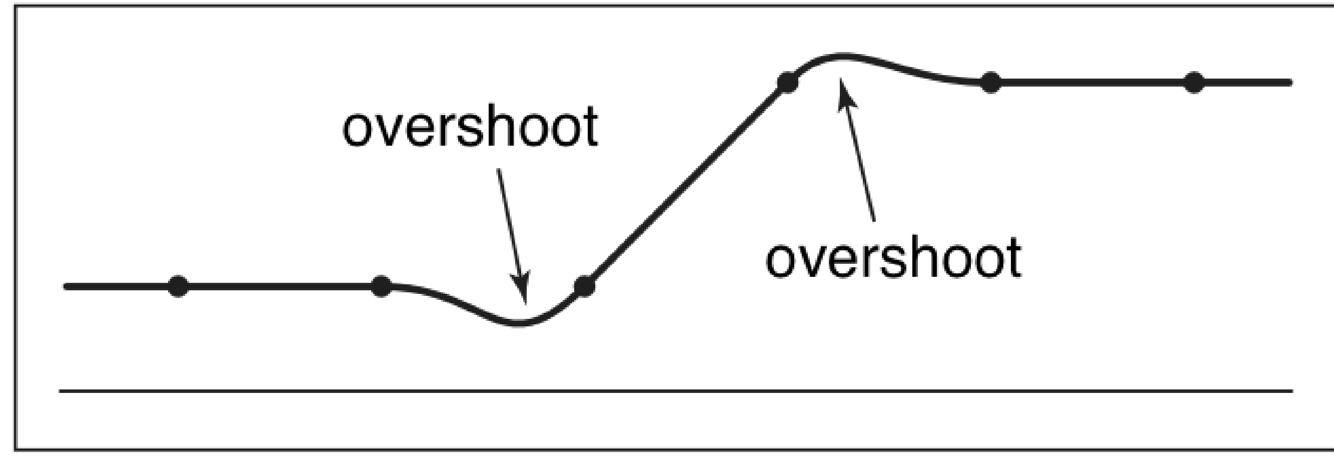


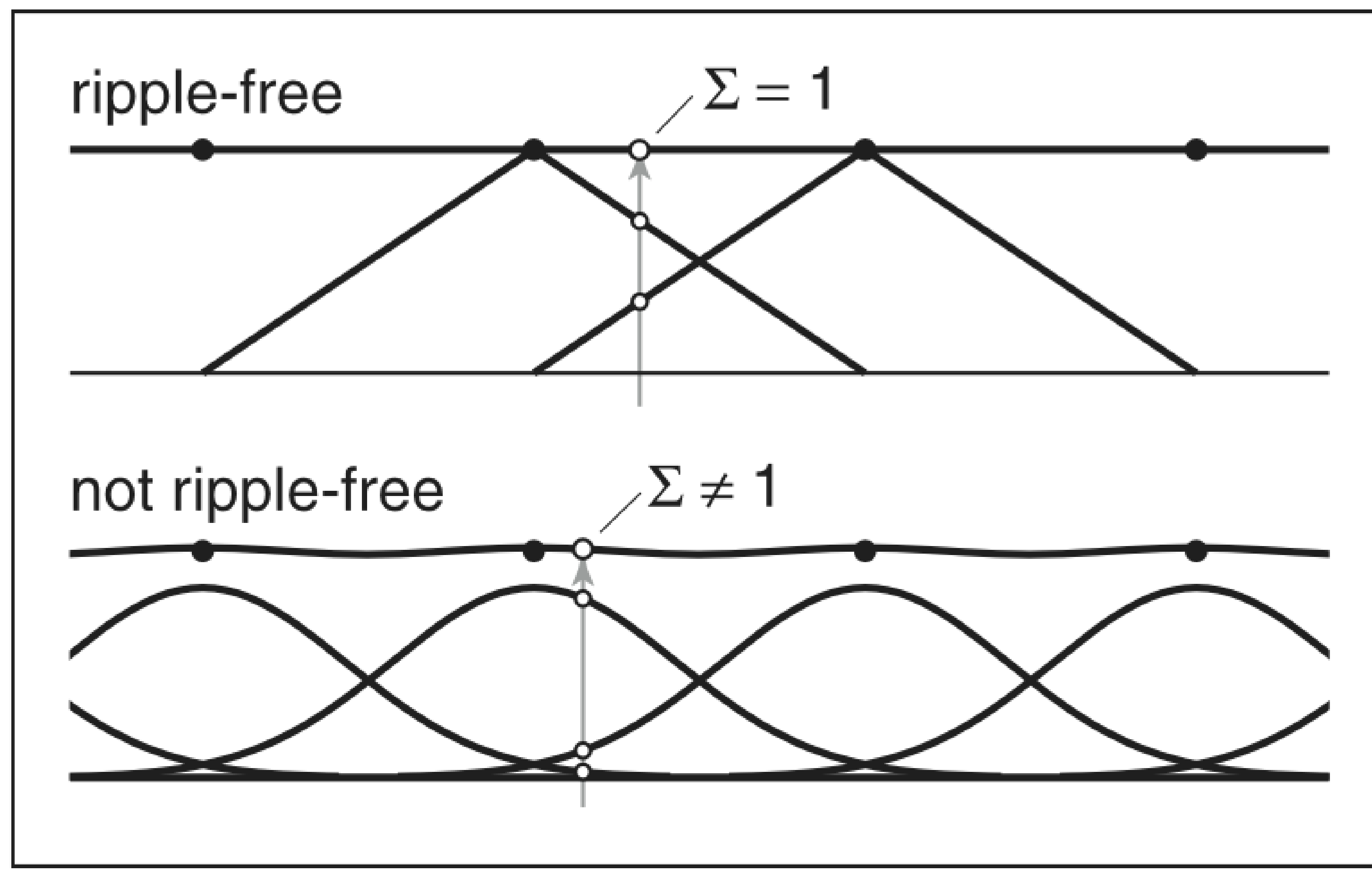
interpolating filter used for reconstruction

Ringing, overshoot, ripples

- Overshoot
  - caused by
    - negative filter values
- Ripples
  - constant in,
    - non-const. out
  - ripple free when:

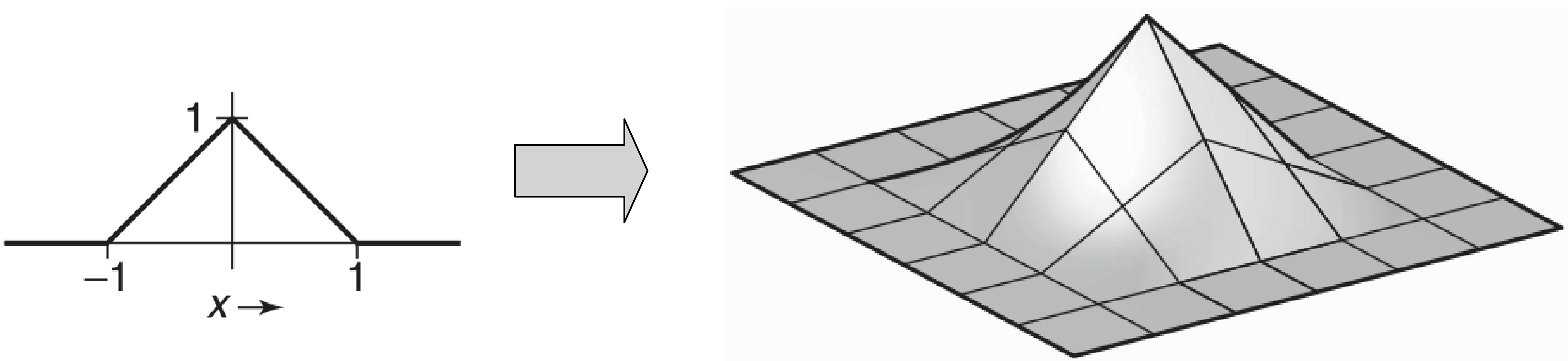
$$\sum_i f(x+i) = 1 \quad \text{for all } x.$$

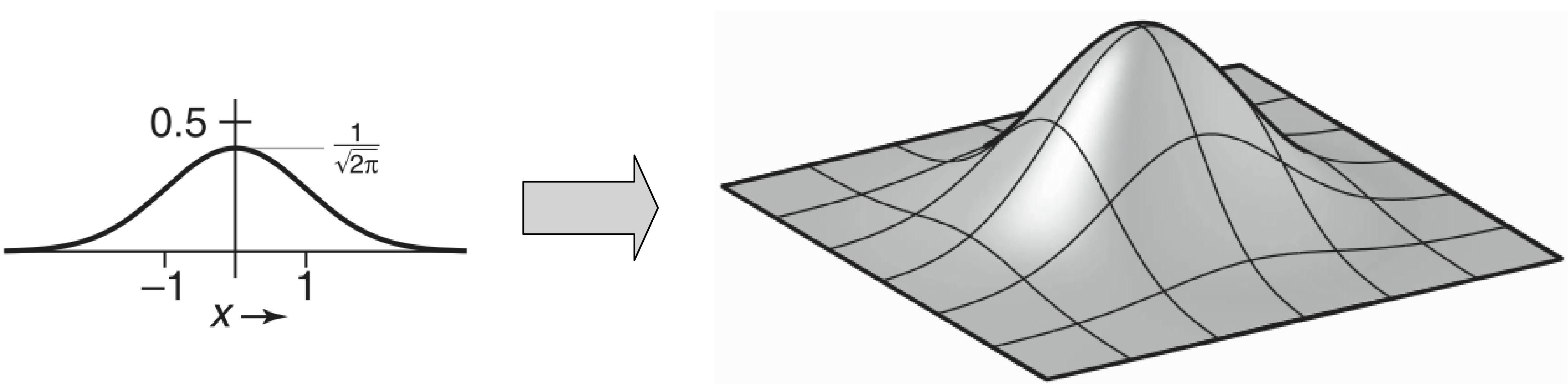




Constructing 2D filters

- Separable filters (most common approach)





Reducing and enlarging

- Very common operation
  - devices have differing resolutions
  - applications have different memory/quality tradeoffs
- Also very commonly done poorly
- Simple approach: drop/replicate pixels
- Correct approach: use resampling

Reducing and enlarging

- Very common operation
  - devices have differing resolutions
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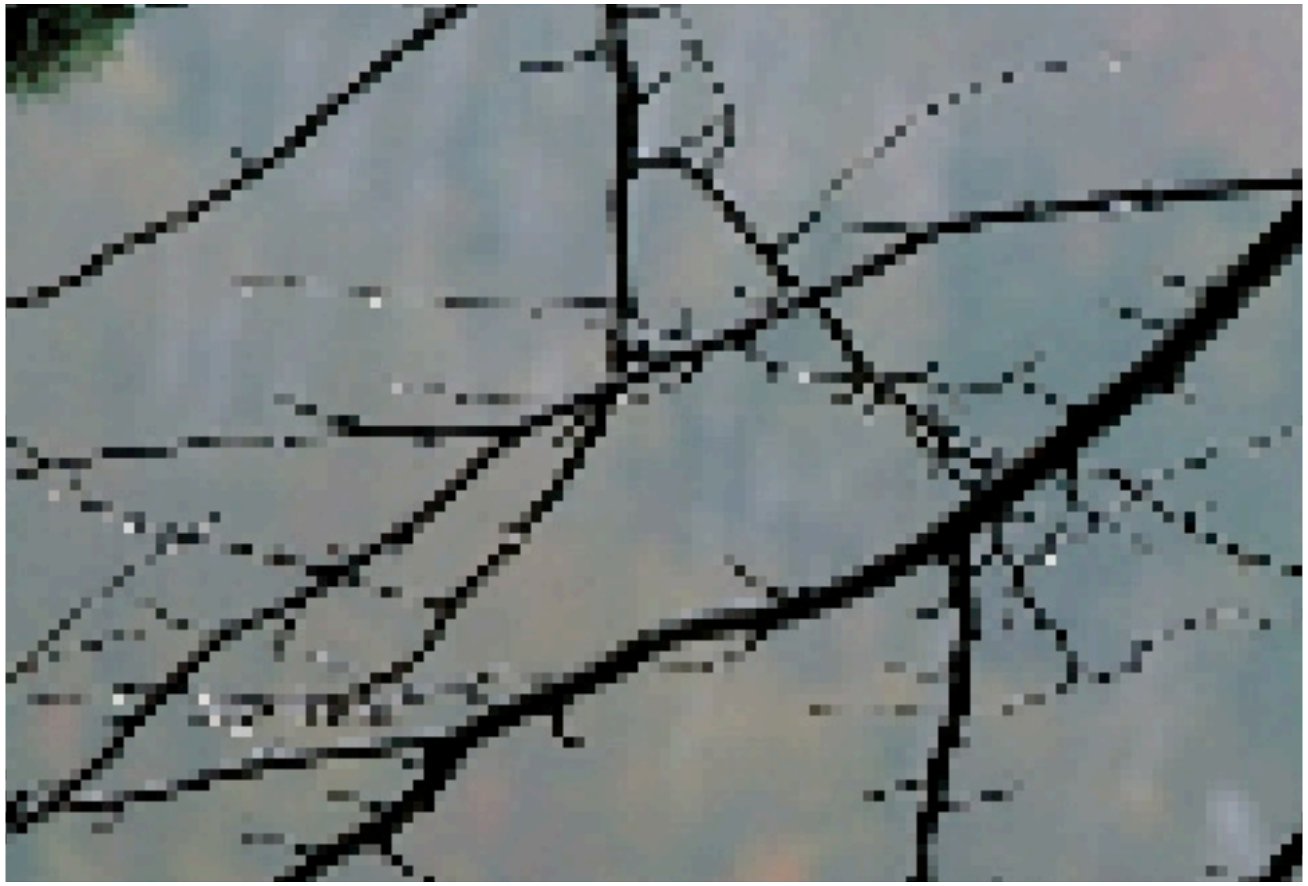
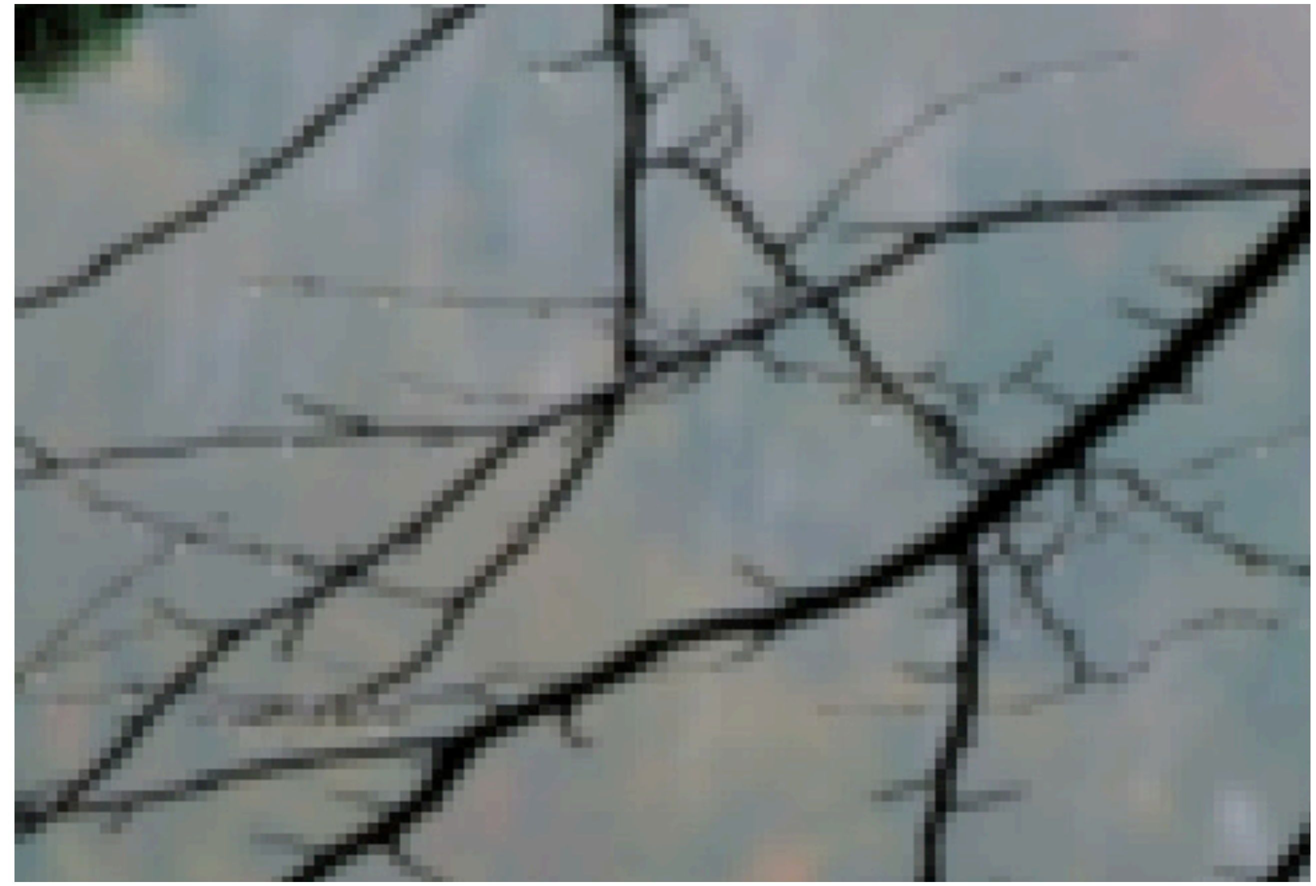
1000 pixel width

[Philip Greenspun]



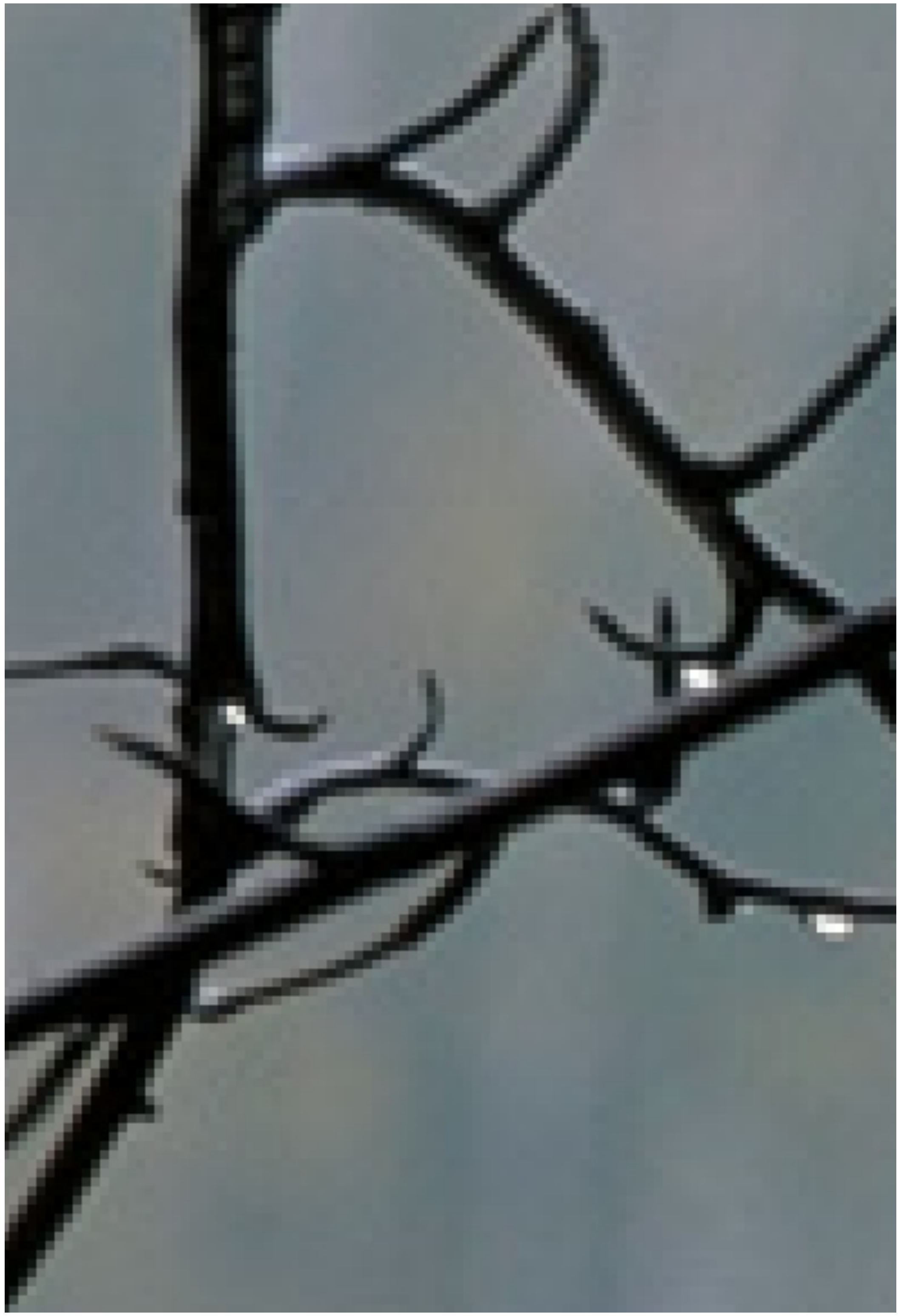


[Philip Greenspun]

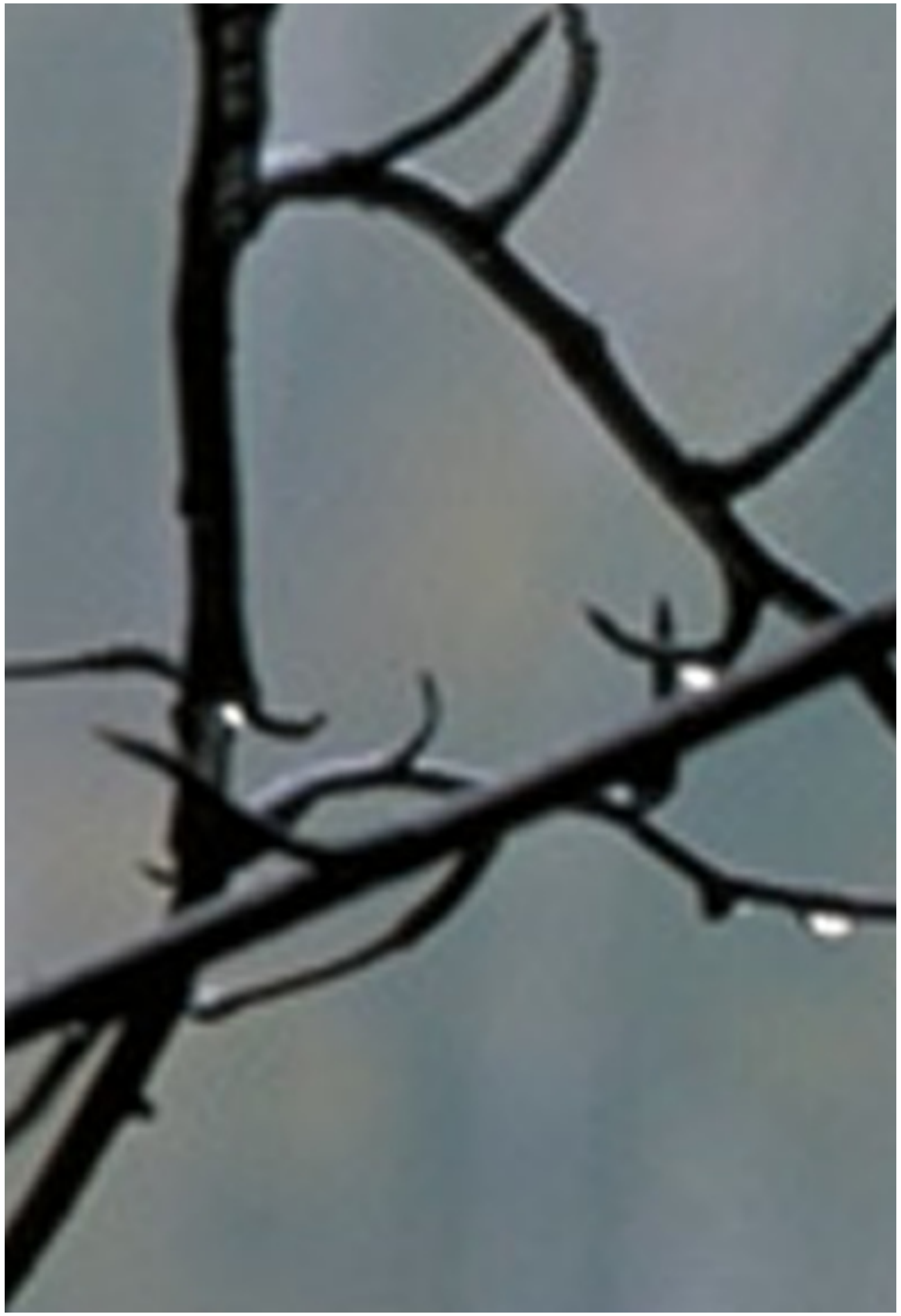
by dropping pixels

gaussian filter

250 pixel width



box reconstruction filter



bicubic reconstruction filter

[Philip Greenspun]

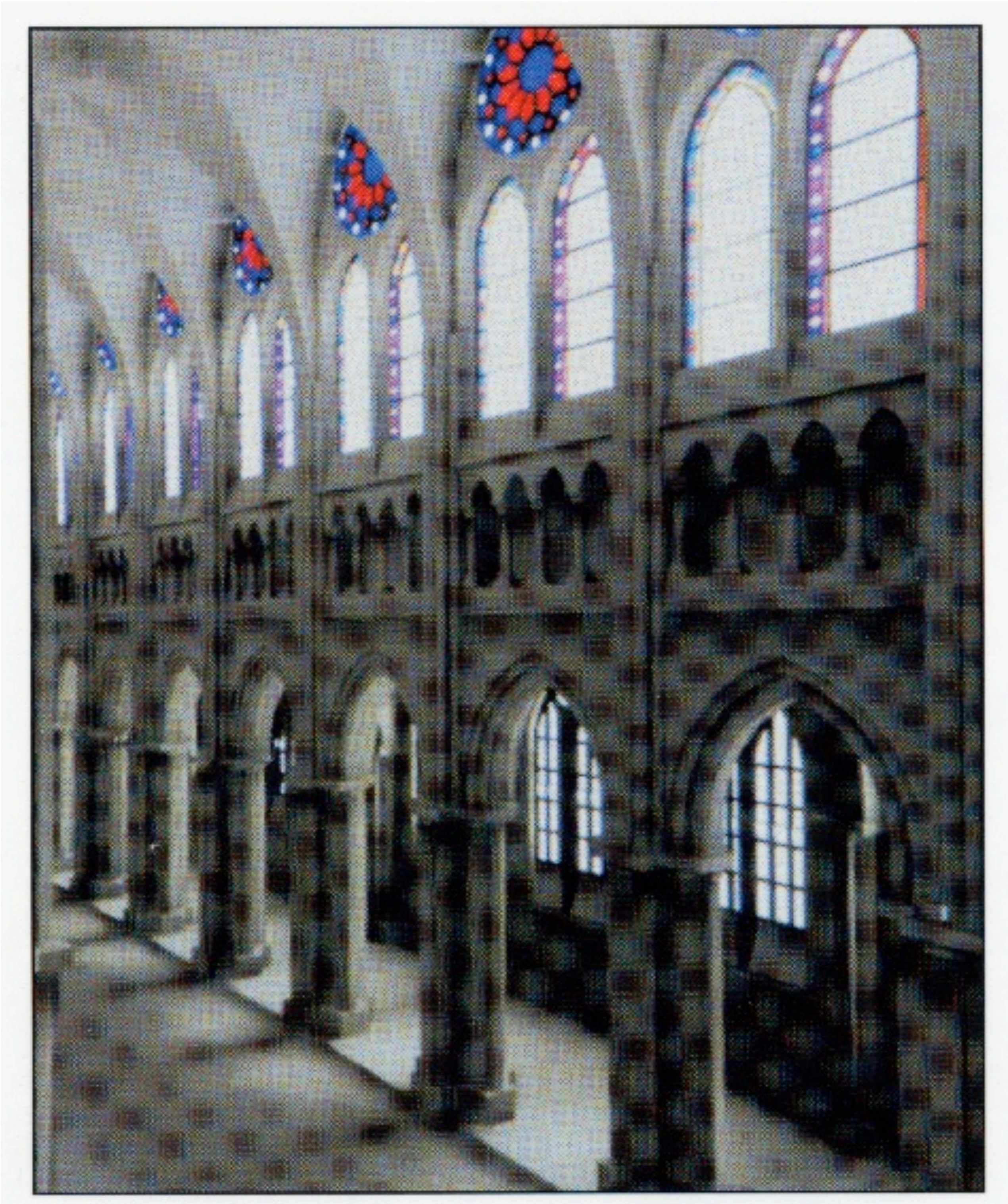
4000 pixel width

Types of artifacts

- Garden variety
  - what we saw in this natural image
  - fine features become jagged or sparkle
- Moiré patterns

[Hearn & Baker cover]

600ppi scan of a color halftone image



by dropping pixels

[Hearn & Baker cover]

gaussian filter

downsampling a high resolution scan

Types of artifacts

- Garden variety

  what we saw in this natural image

  fine features become jagged or sparkle

- Moiré patterns

  caused by repetitive patterns in input

  produce large-scale artifacts; highly visible

- These artifacts are *aliasing* just like in the audio example earlier

- How do I know what filter is best at preventing aliasing?

  practical answer: experience

  theoretical answer: there is another layer of cool math behind all this

  - based on Fourier transforms
  - provides much insight into aliasing, filtering, sampling, and reconstruction

Checkpoint

- Want to formalize sampling and reconstruction
  - define impulses
  - then we can talk about S&R with only one datatype
- Define Fourier transform
- Destination: explaining how aliases leak into result

Mathematical model

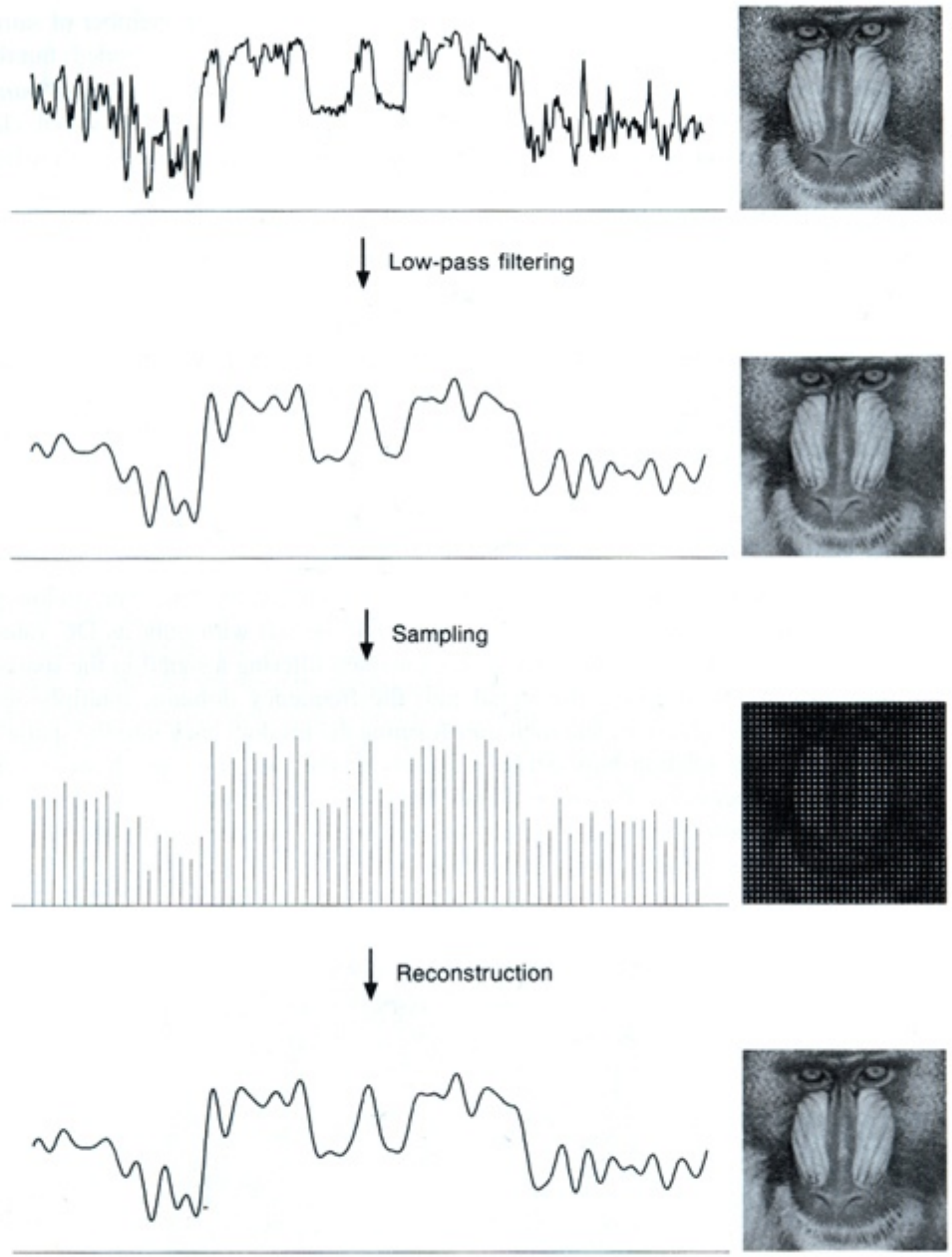
- We have said sampling is storing the values on a grid
- For analysis it's useful to think of the sampled representation in the same space as the original
  - I'll do this using *impulse functions* at the sample points

Impulse function

- A function that is confined to a very small interval
  - but still has unit integral
  - really, the limit of a sequence of ever taller and narrower functions
  - also called Dirac delta function
- Key property: multiplying by an impulse selects the value at a point
  - Defn via integral
- Impulse is the identity for convolution
  - "impulse response" of a filter

Sampling & recon. reinterpreted

- Start with a continuous signal
- Convolve it with the sampling filter
- Multiply it by an impulse grid
- Convolve it with the reconstruction filter

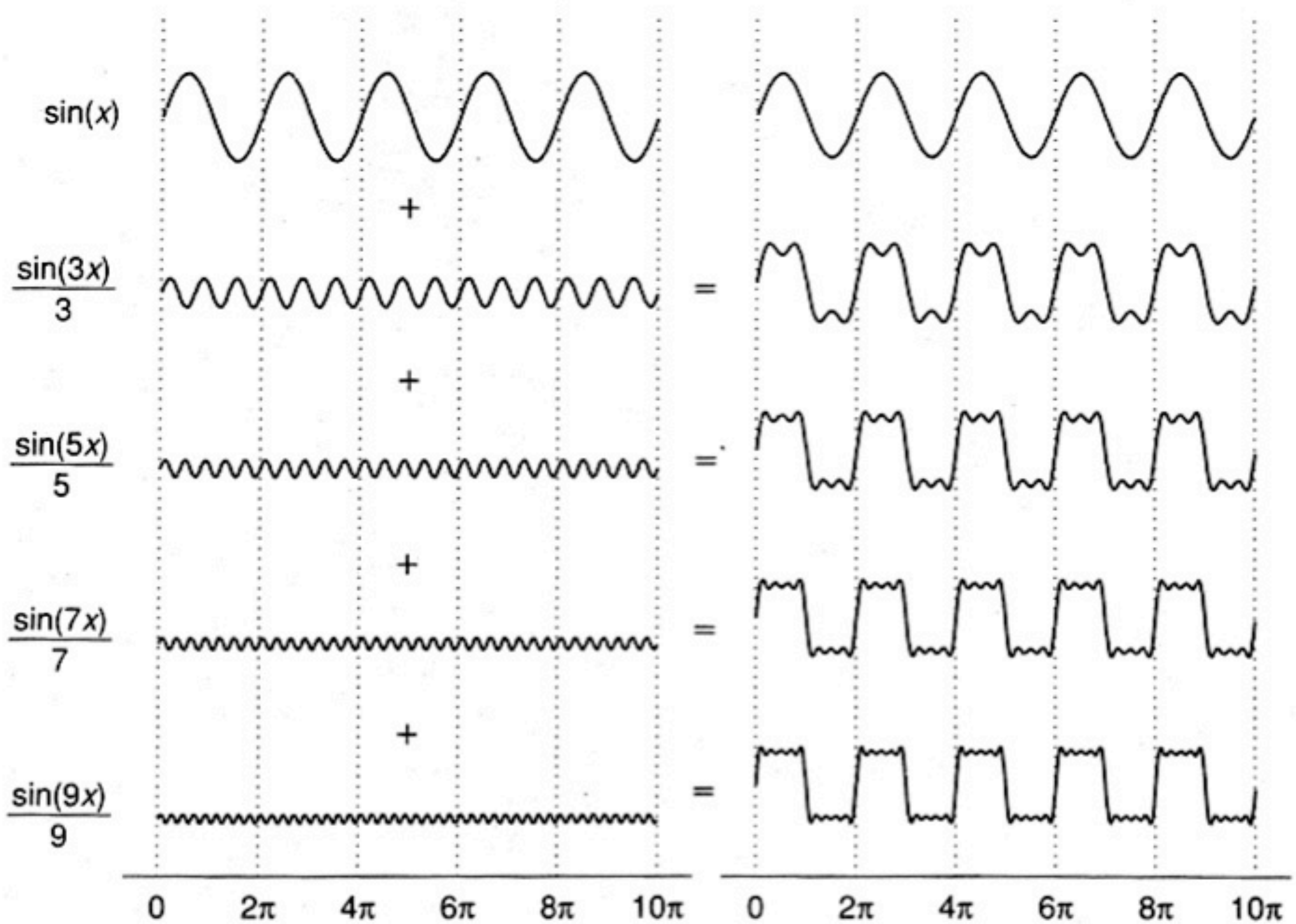

Low-pass filtering
Sampling
Reconstruction

Checkpoint

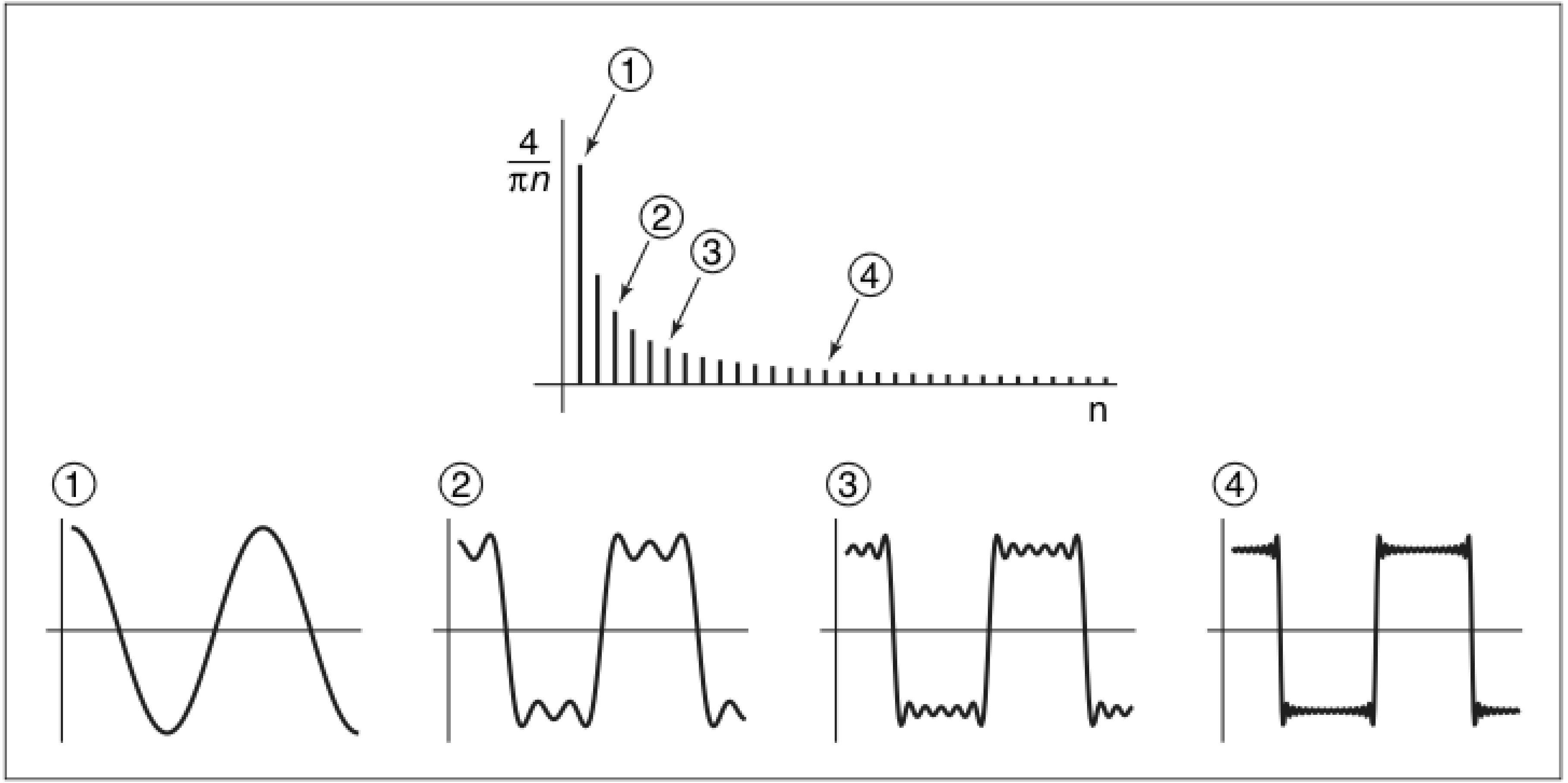
- Formalized sampling and reconstruction
  - used impulses with multiplication and convolution
  - can talk about S&R with only one datatype
- Define Fourier transform
- Destination: explaining how aliases leak into result

Fourier series

- Probably familiar idea of adding up sines and cosines to approximate a periodic function



Fourier series

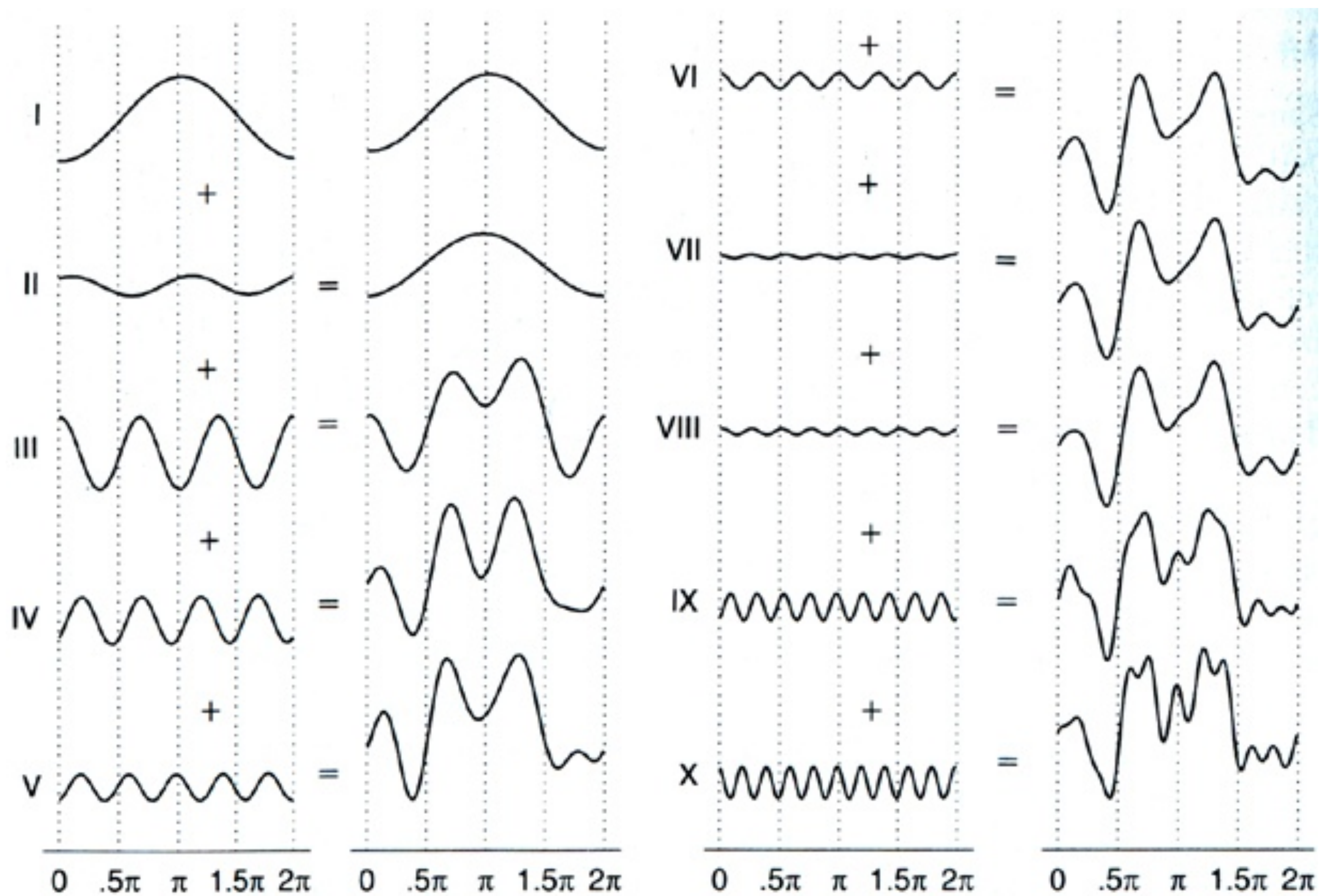


Fourier transform

- Like Fourier series but for aperiodic functions
  - Fourier series: only multiples of base frequency
- Fourier transform: let period go to infinity
  - eventually all frequencies are needed
  - result: countable sum turns into integral

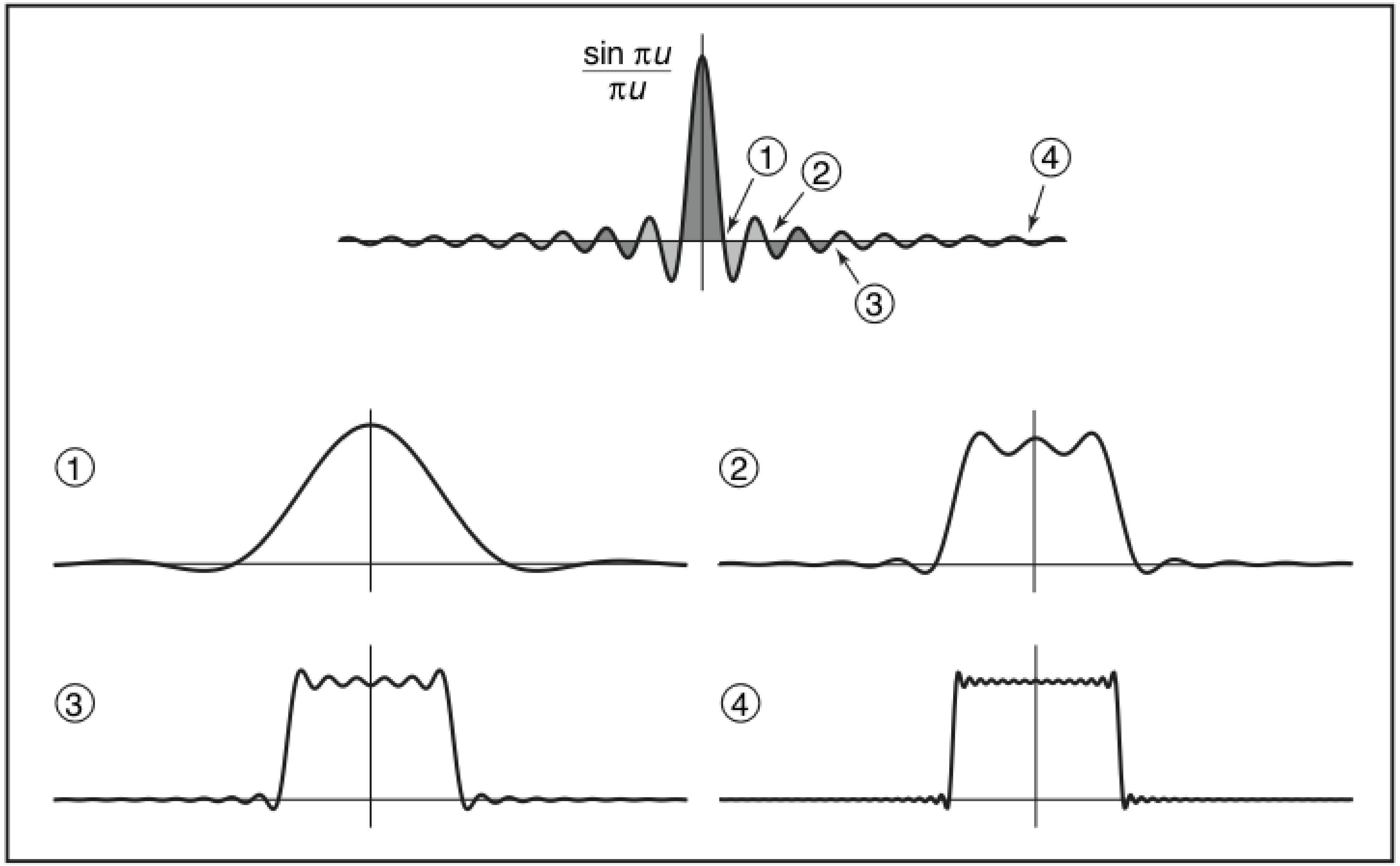
The Fourier transform

- Any function on the real line can be represented as an infinite sum of sine waves



[F]

The Fourier transform



The Fourier transform

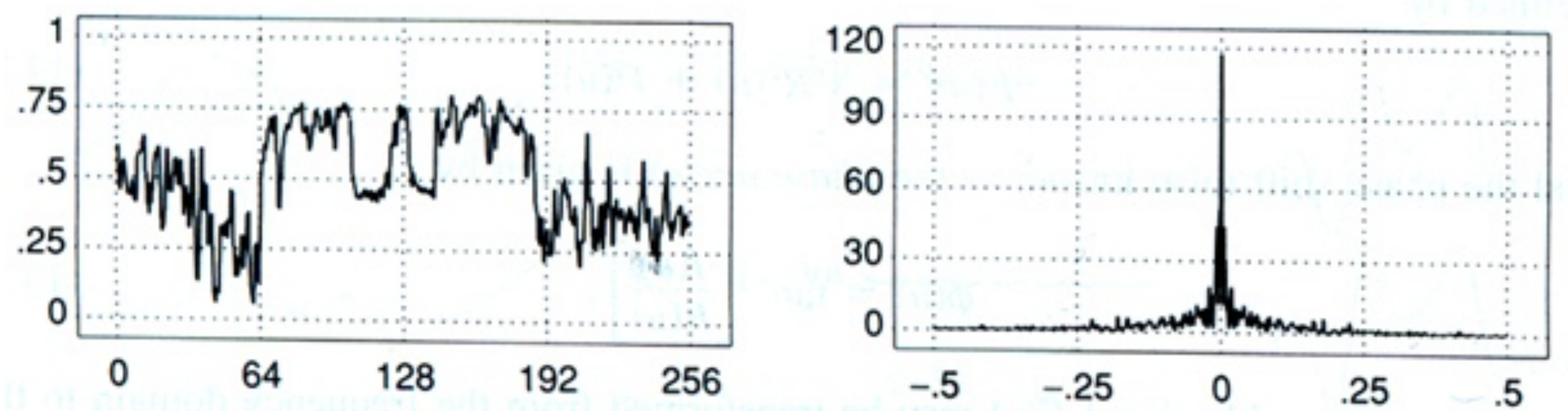
- The coefficients of those sine waves form a continuous function of frequency
- That function, which has the same datatype as the first one, is the Fourier transform.

$$F(u) = \int_{-\infty}^{\infty} f(x)(\cos 2\pi ux - i \sin 2\pi ux)dx$$

- Phase encoded in complex number

Fourier transform properties

- F.T. is its own inverse (just about)
- Frequency space is a dual representation
  - amplitude known as "spectrum"

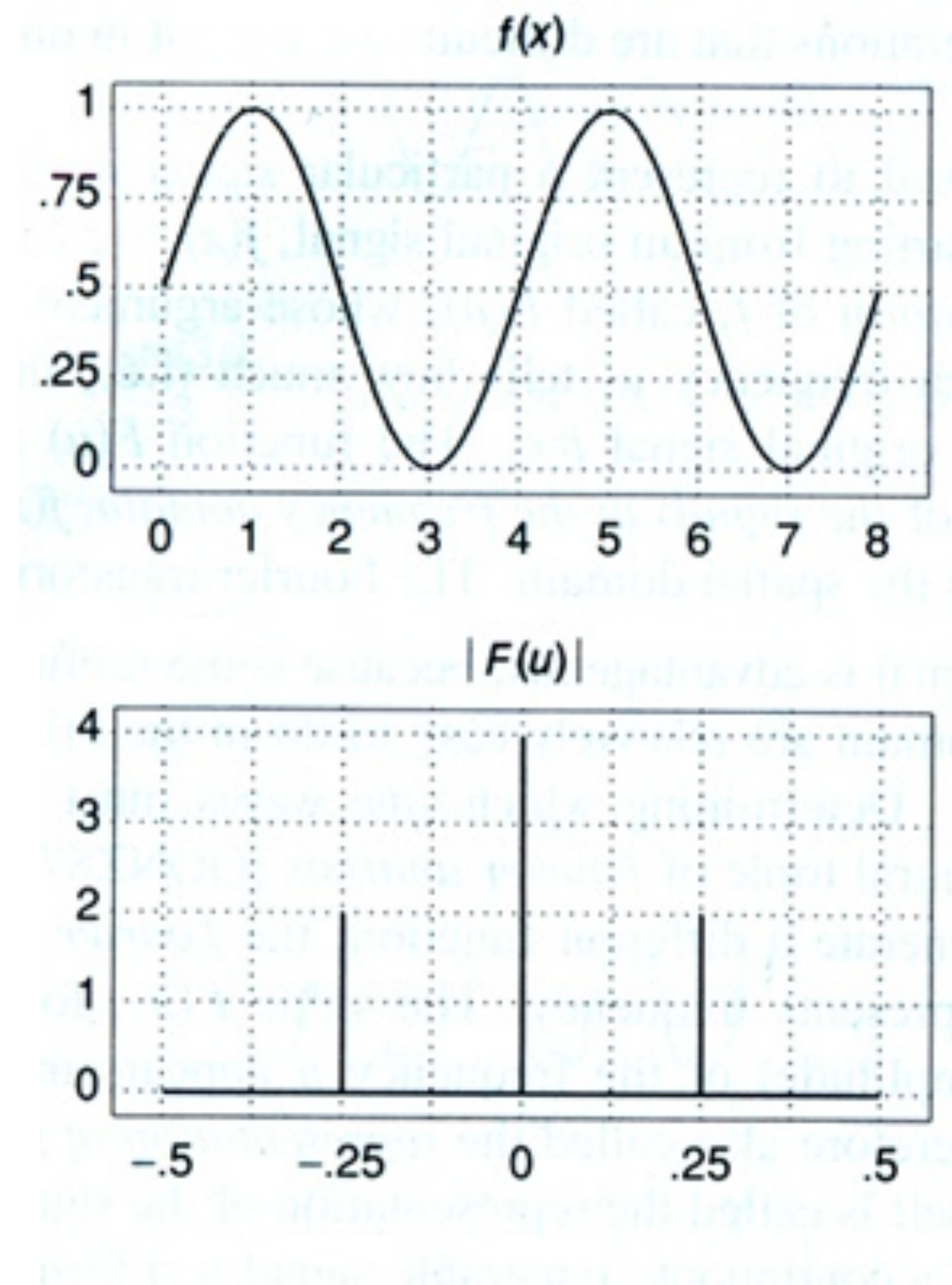


(c)

[FvDFH fig.14.15 / Wolberg]

Fourier pairs

- sinusoid — impulse pair
- box — sinc
- tent — sinc^2
- bspline — sinc^4
- gaussian — gaussian
  (inv. width)
- imp. grid — imp. grid
  (1/d spacing)



[FvDFH fig.14.25 / Wolberg]

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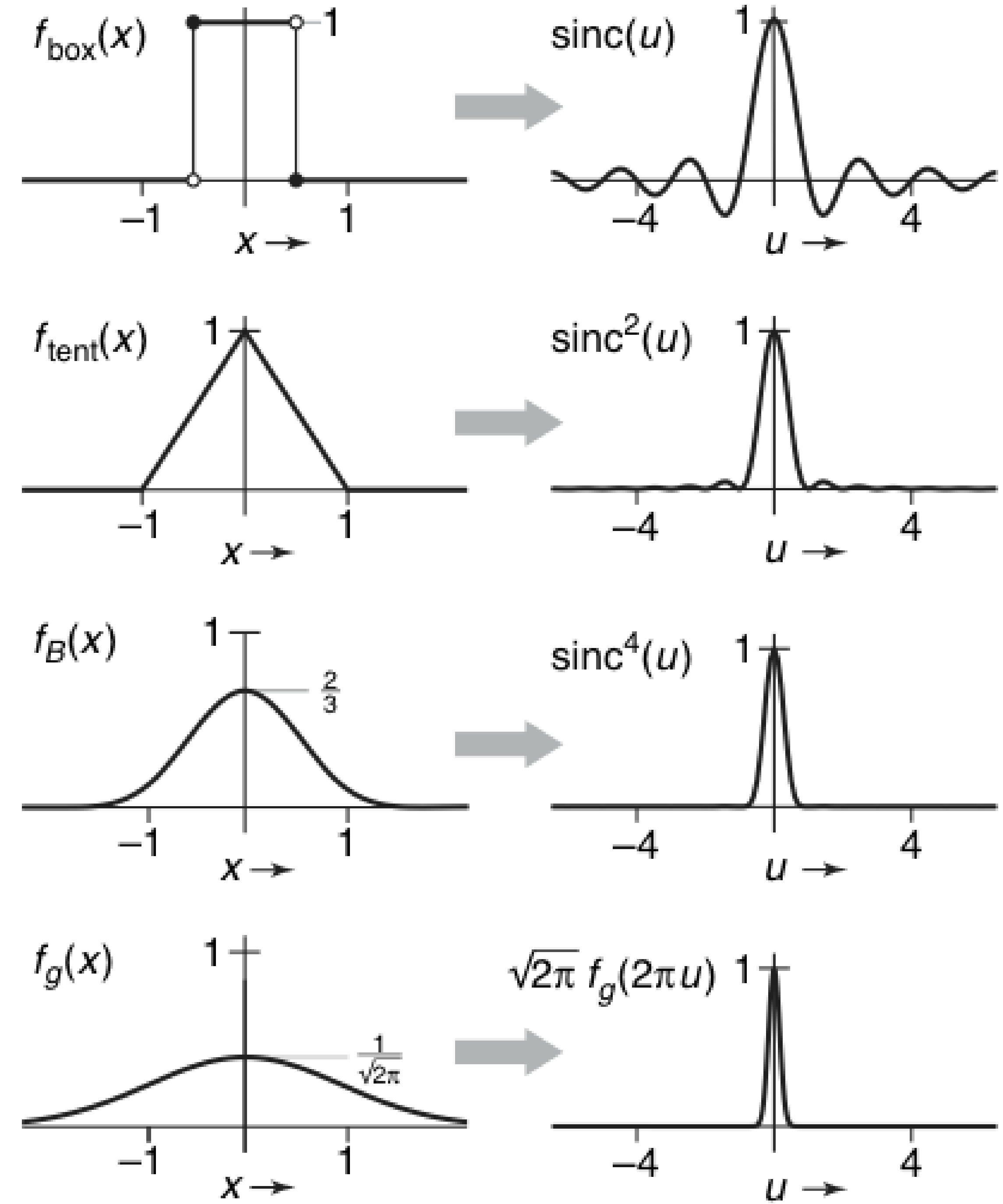

[FvDFH fig.14.25 / Wolberg]

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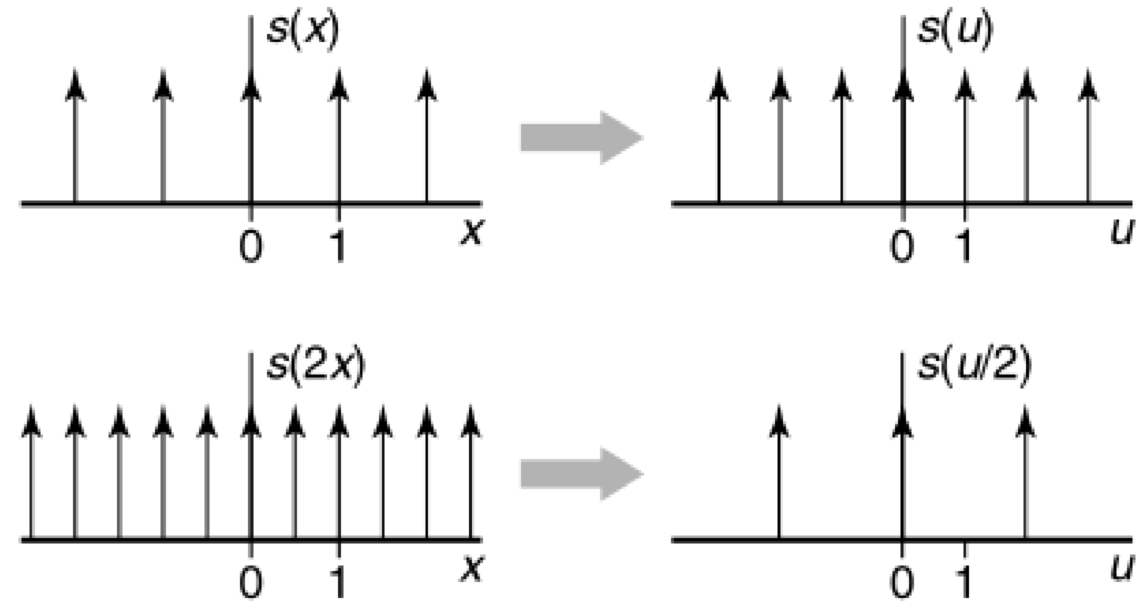
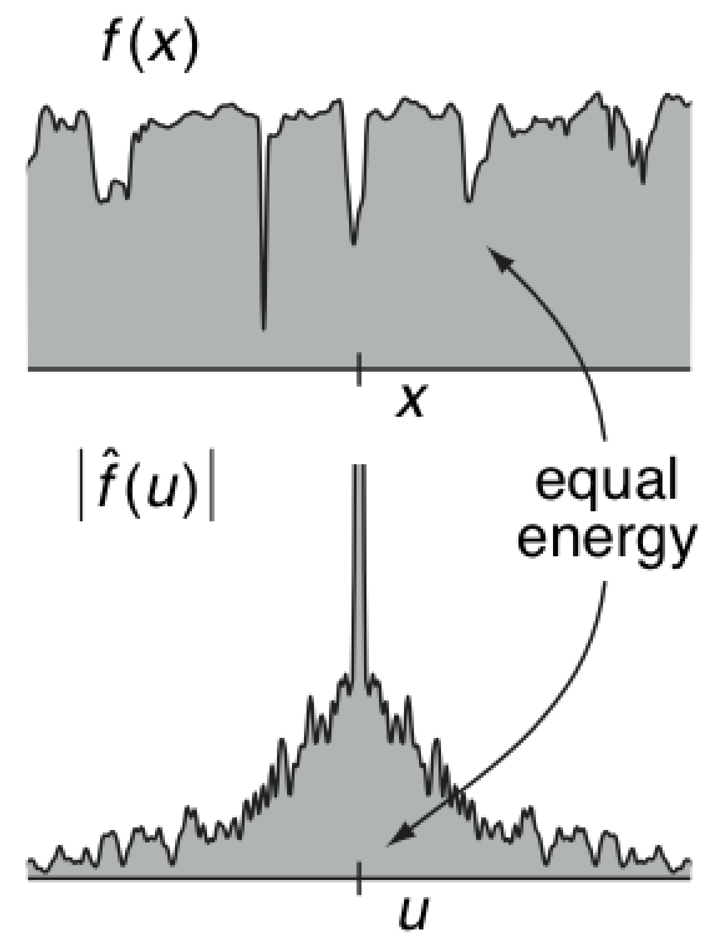


[FvDFH fig.14.25 / Wolberg]

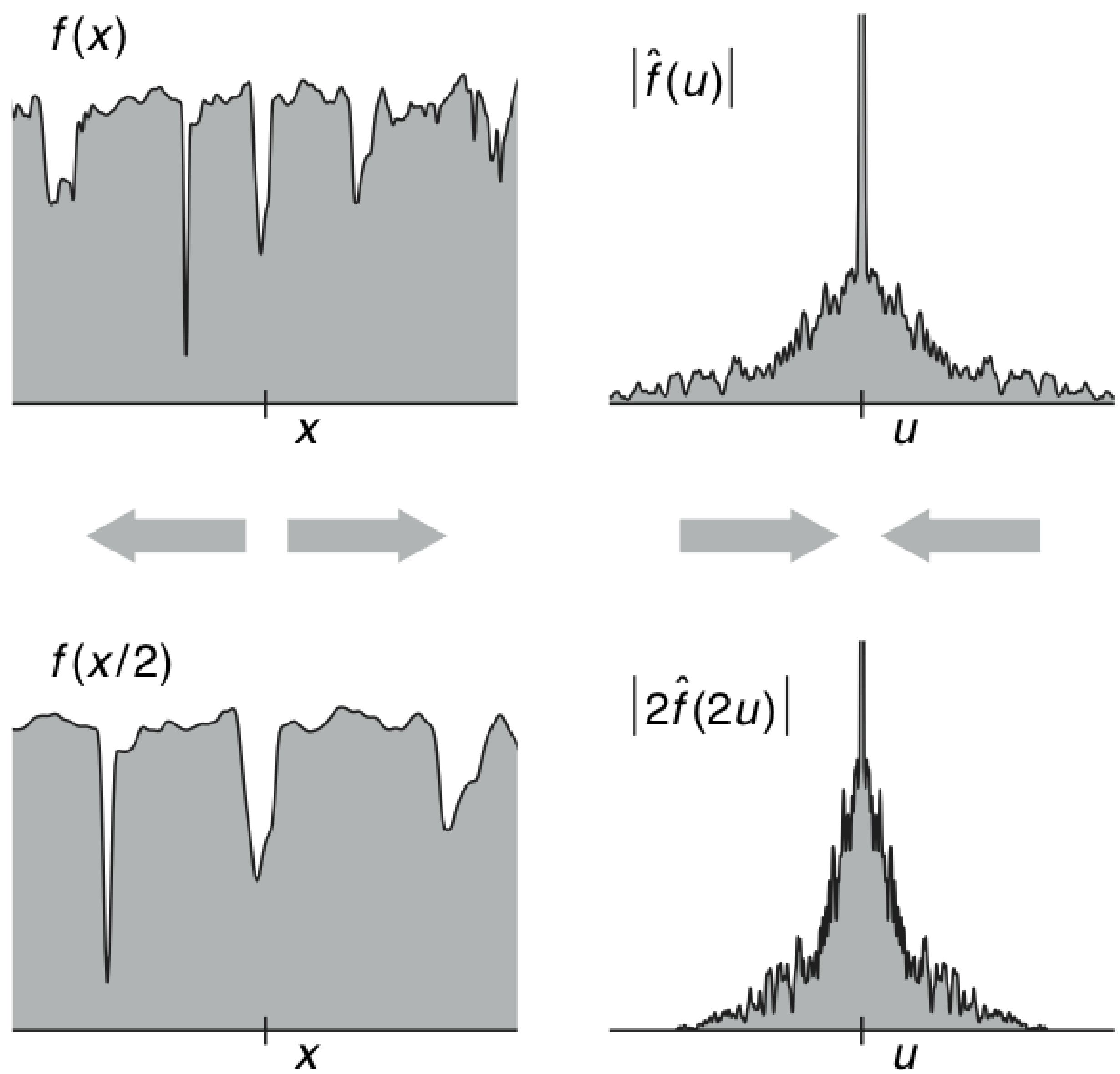
More Fourier facts

- F.T. preserves energy
  - That is, the squared integral
- DC component (average value)
  - It shows up at F(0)



More Fourier facts

- Dilation (stretching/squashing)
  - Results in inverse dilation in F.T.



Convolution and multiplication

- They are dual to one another under F.T.

$$\mathcal{F}\{f * g\}(u) = F(u)G(u)$$
$$\mathcal{F}\{fg\}(u) = (F * G)(u)$$

- Lowpass filters
  - Most of our "blurring" filters have most of their F.T. at low frequencies
  - Therefore they attenuate higher frequencies

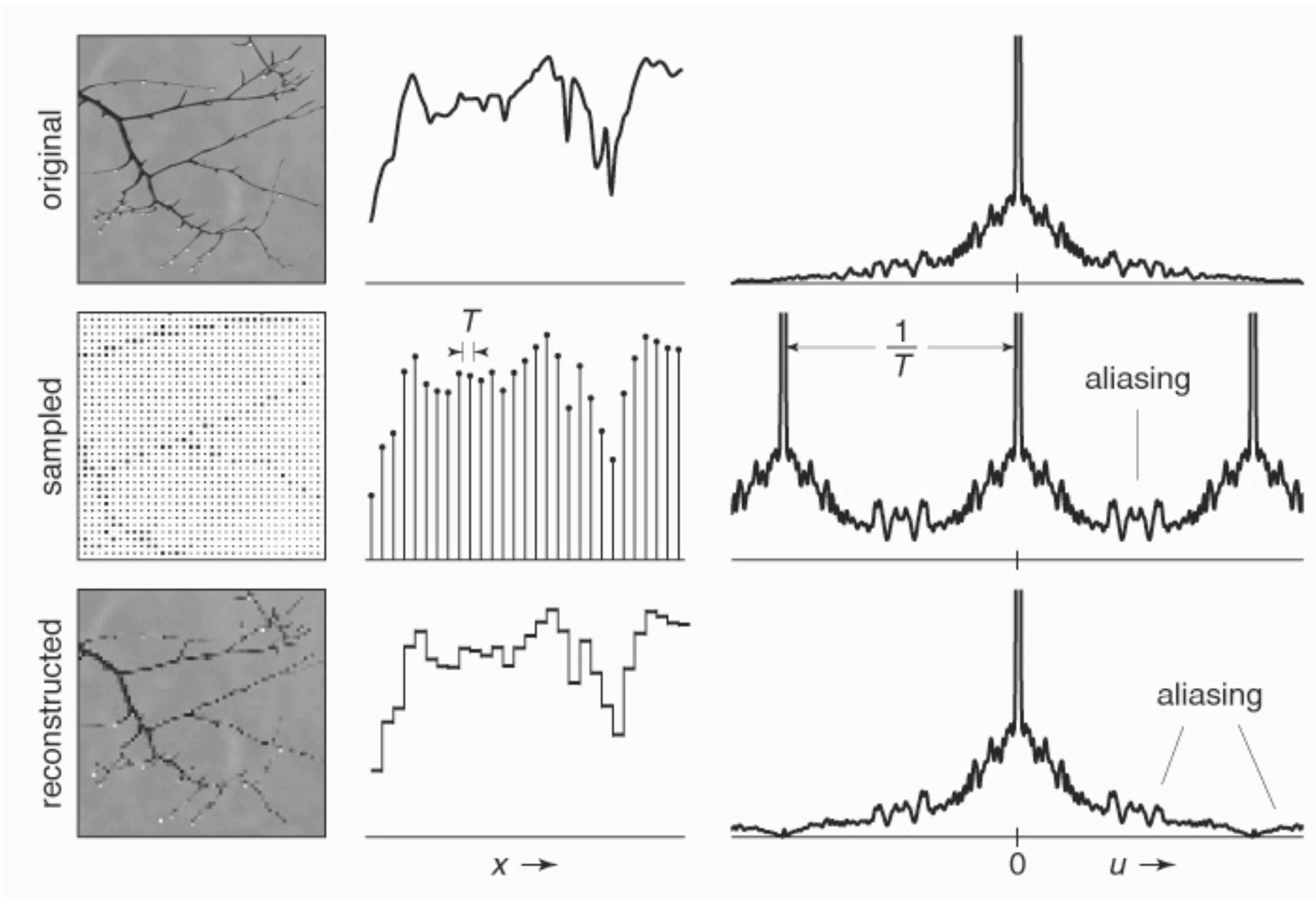
Checkpoint

- Formalized sampling and reconstruction
  - used impulses with multiplication and convolution
- Can talk about S&R with only one datatype
- Defined Fourier transform
  - alternate representation for functions
  - turns convolution, which seems hard, into multiplication, which is easy
- Destination: explaining how aliases leak into result

Sampling and reconstruction in F.T.

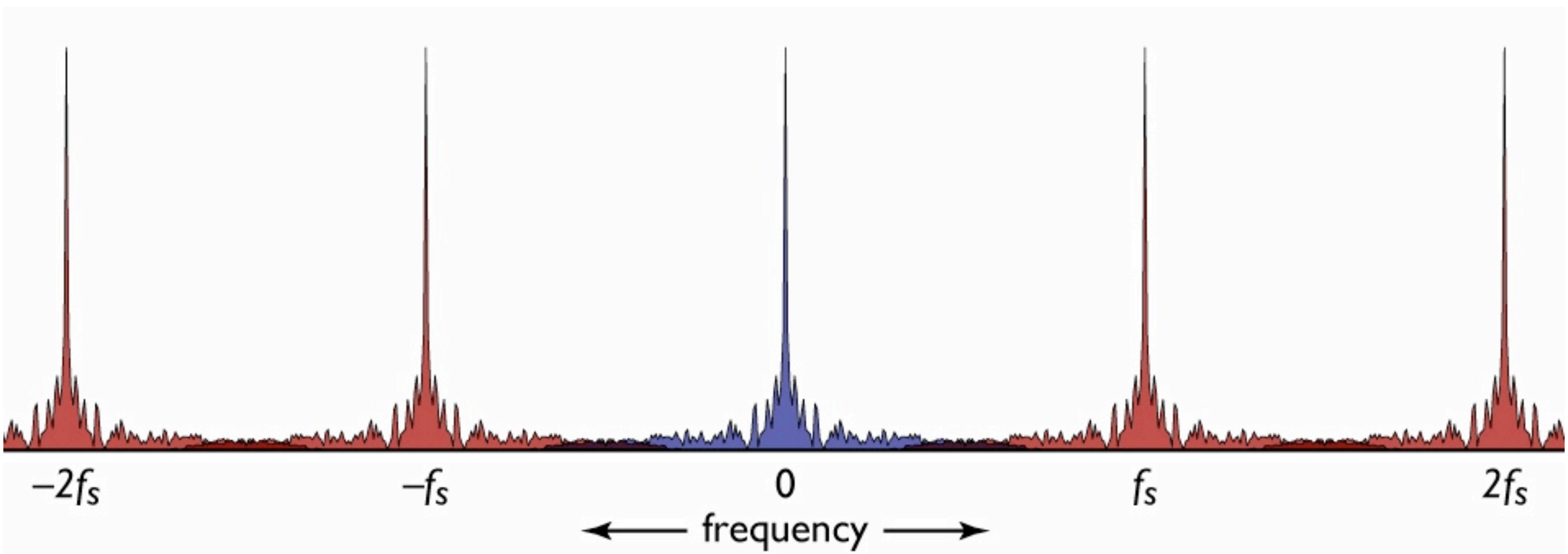
- Look at our sampling/reconstruction formulation in Fourier domain
  - Convolve with filter = remove high frequencies
  - Multiply by impulse grid = convolve with impulse grid
    - that is, make a bunch of copies
  - Convolve with filter = remove extra copies
  - Left with approximation of original
    - but filtered a couple of times

Aliasing in sampling/reconstruction



Aliasing in sampling

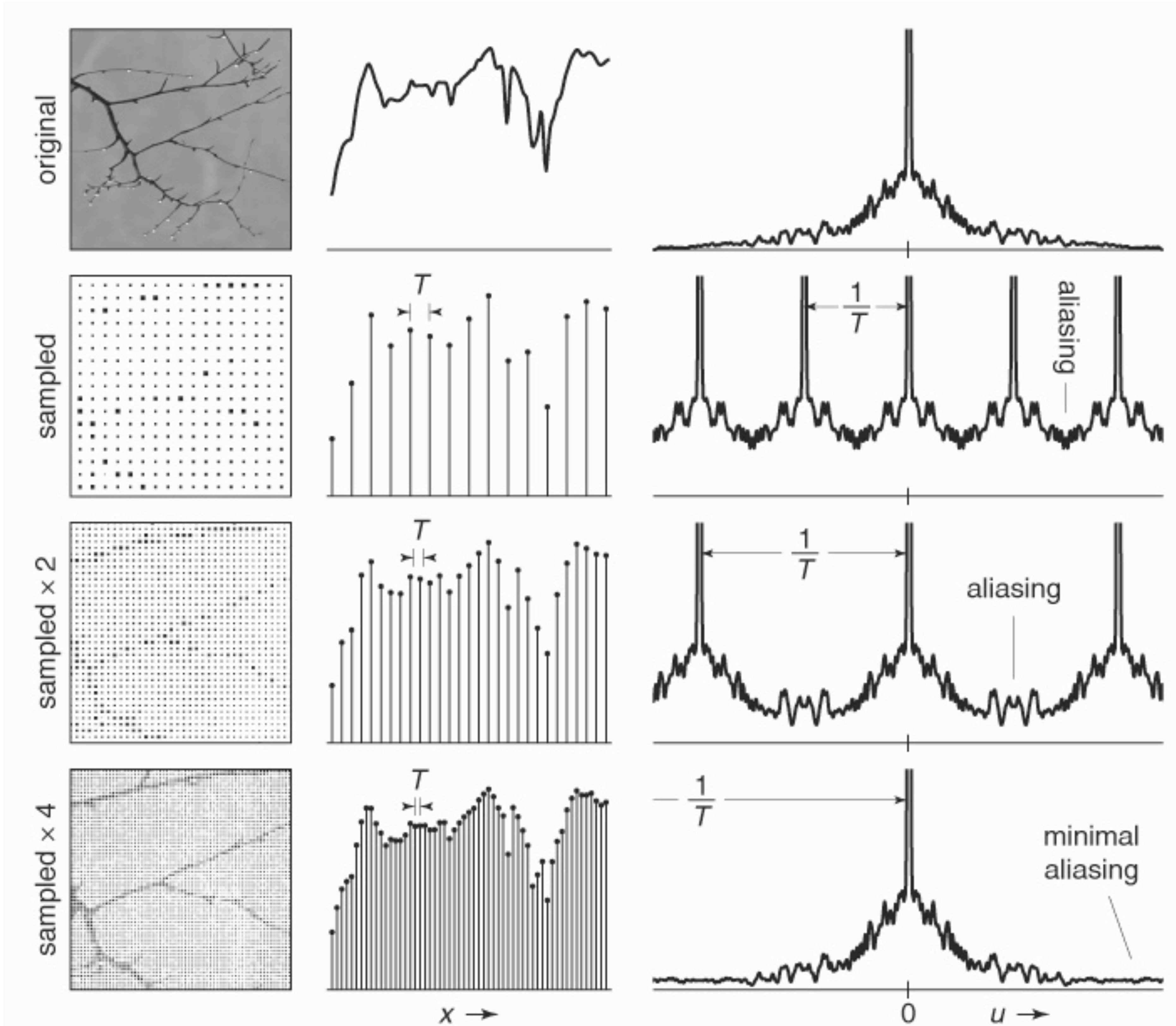
- If sampling filter is not adequate, spectra will overlap
- No way to fix once it's happened
  - can only use drastic reconstruction filter to eliminate
- Nyquist criterion



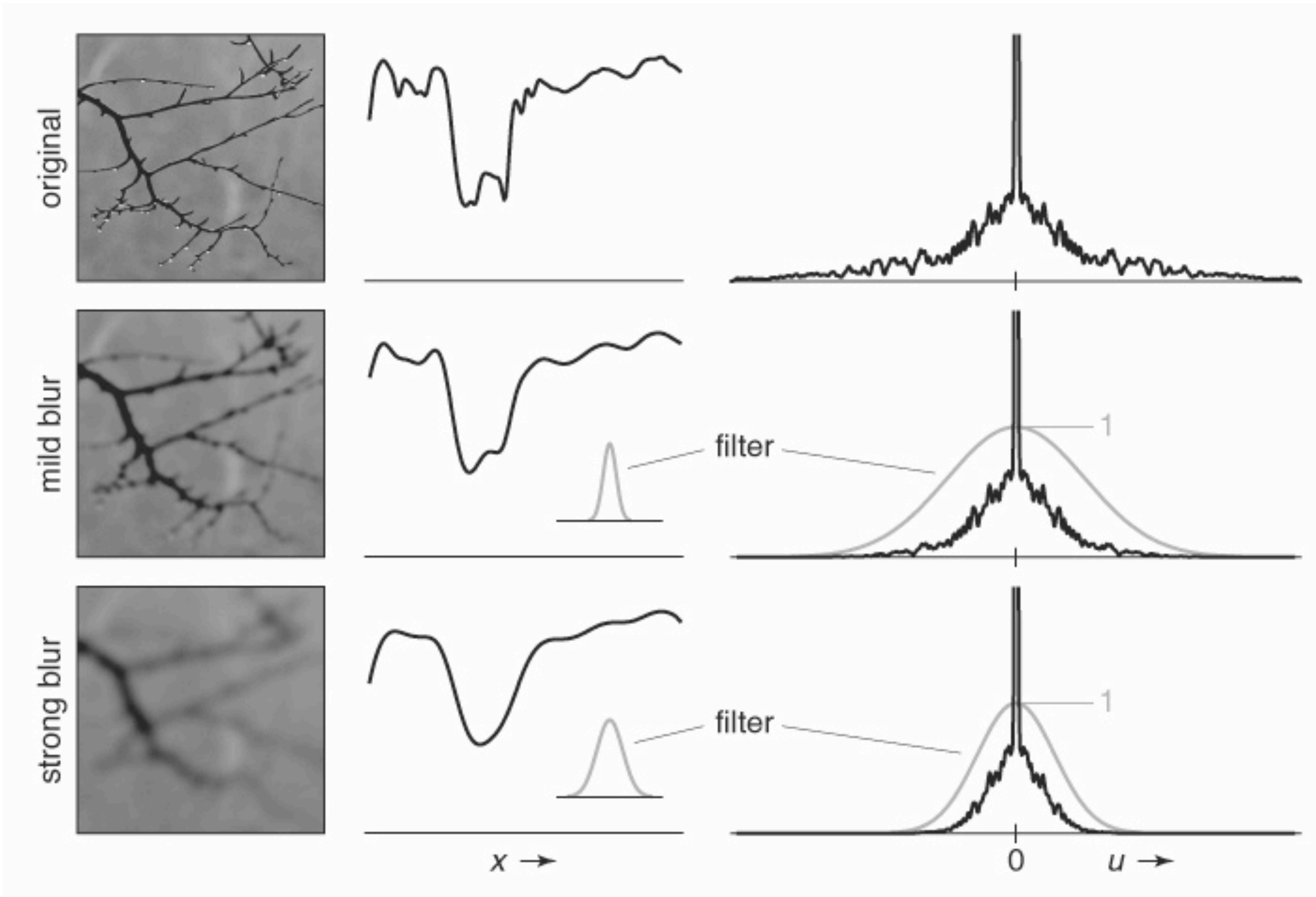
Preventing aliasing in sampling

- Use high enough sample frequency
  - works when signal is *band limited*
  - sample rate 2 * (highest freq.) is enough to capture all details
- Filter signal to remove high frequencies
  - make the signal band limited
  - remove frequencies above 0.5 * (sample freq.) (Nyquist)

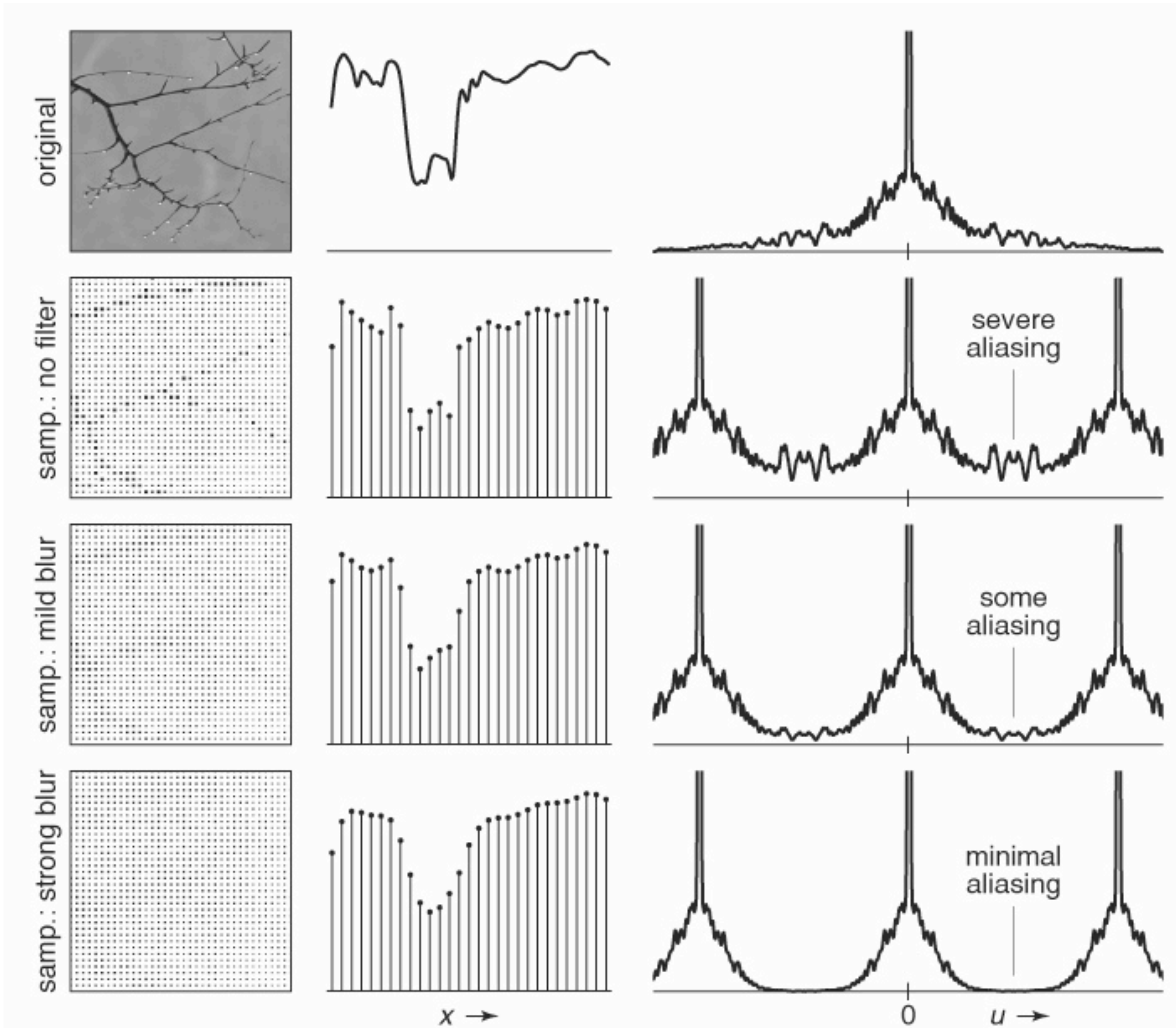
Effect of sample rate on aliasing



Smoothing (lowpass filtering)

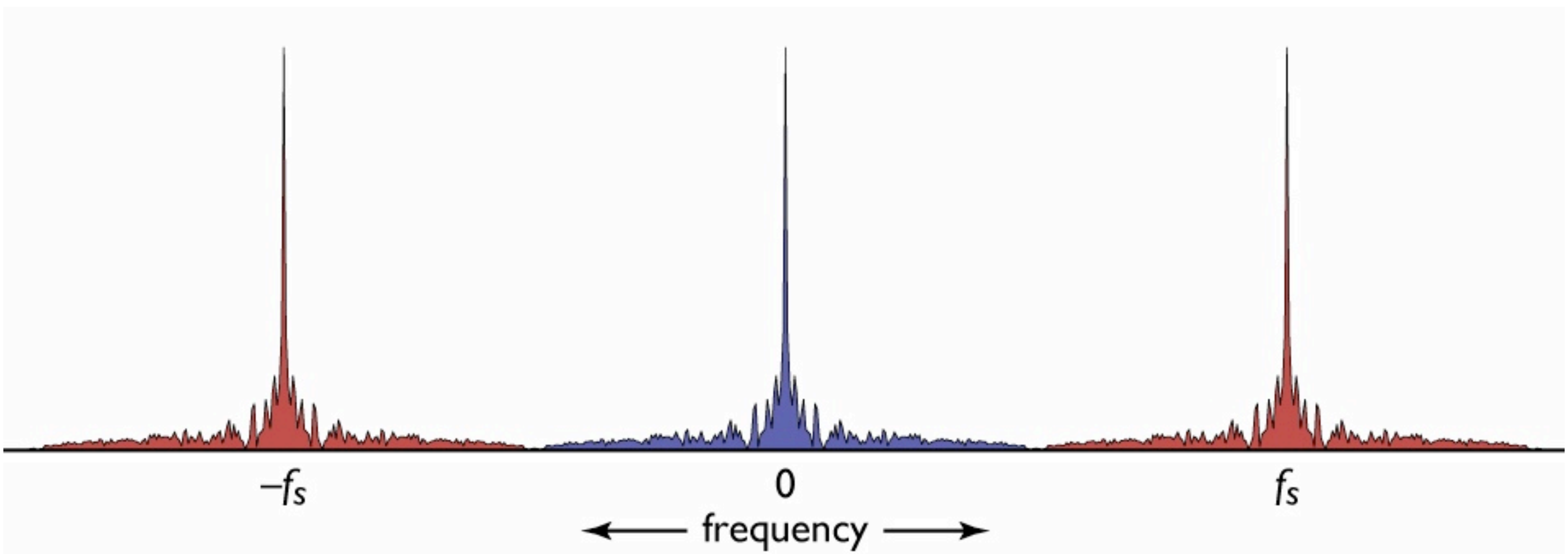


Effect of smoothing on aliasing



Aliasing in reconstruction

- If reconstruction filter is inadequate, will catch alias spectra
- Result: high frequency alias components
- Can happen even if sampling is ideal



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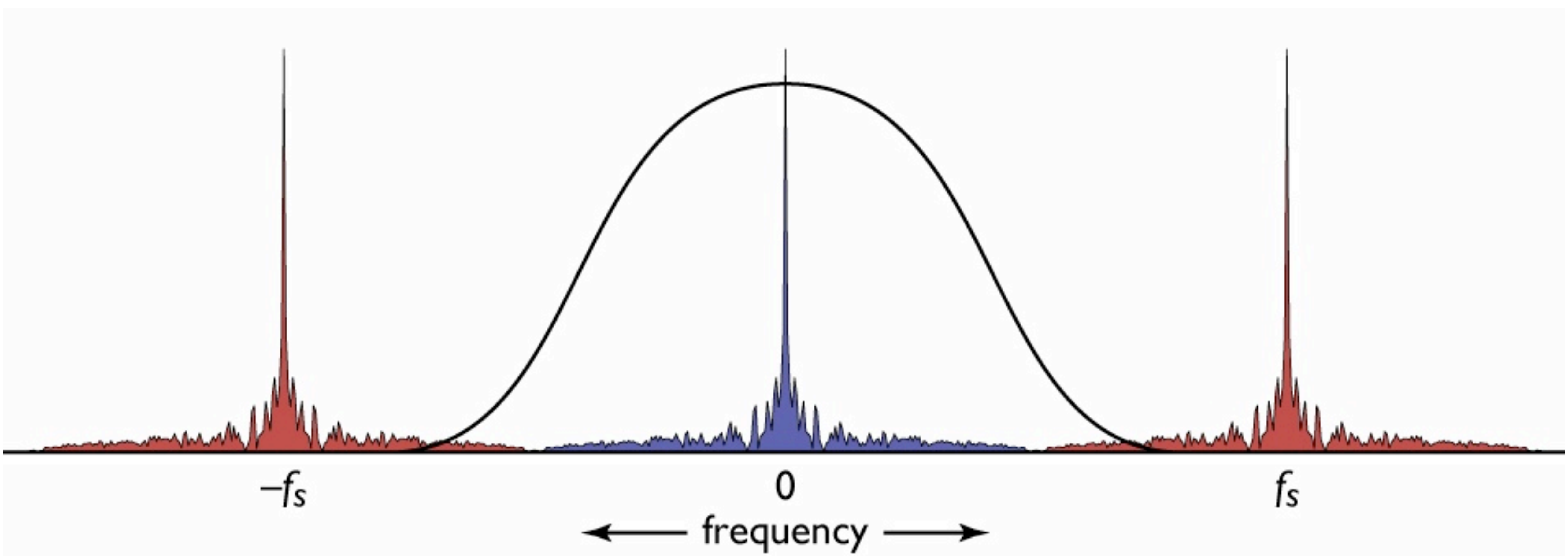
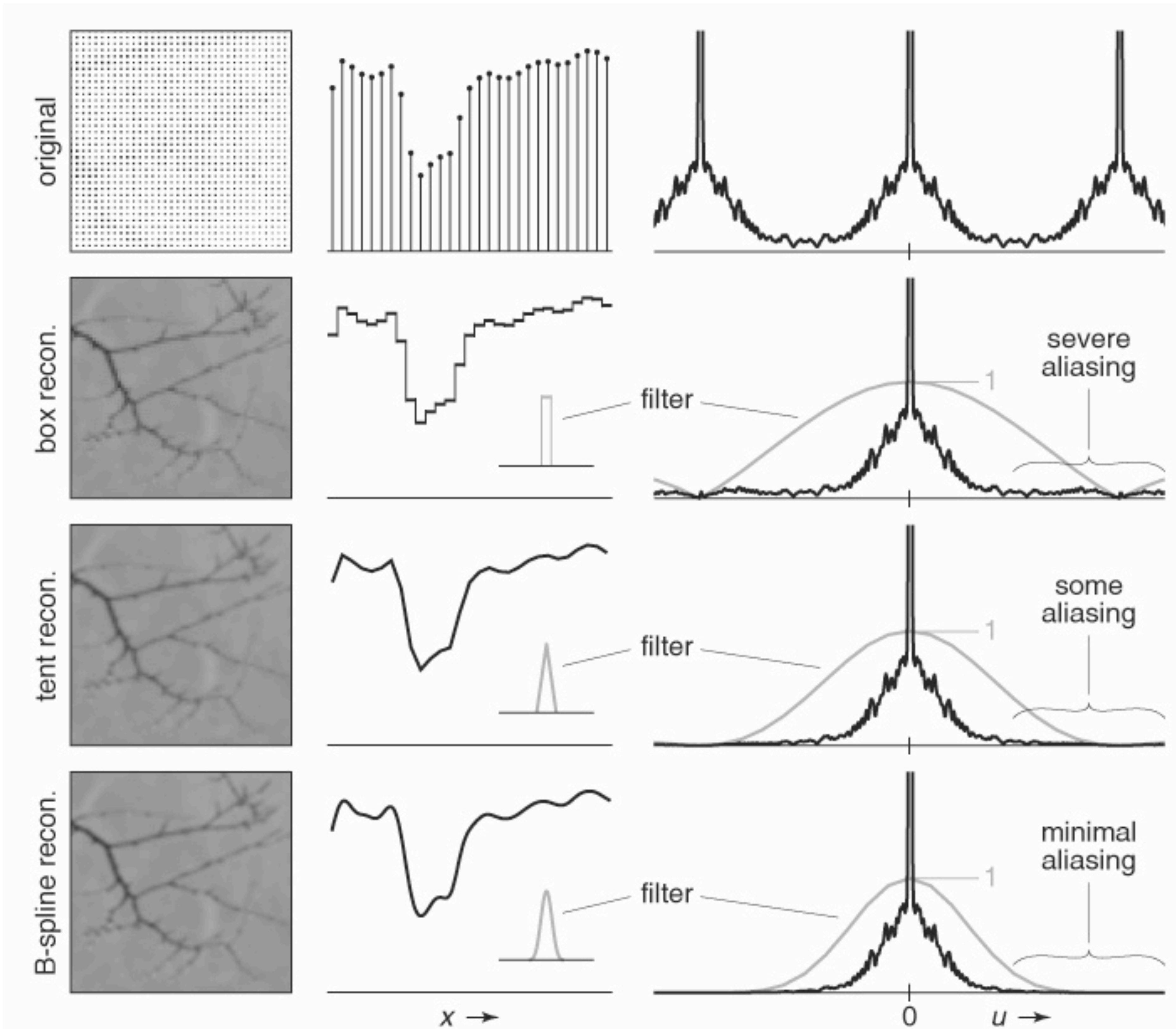
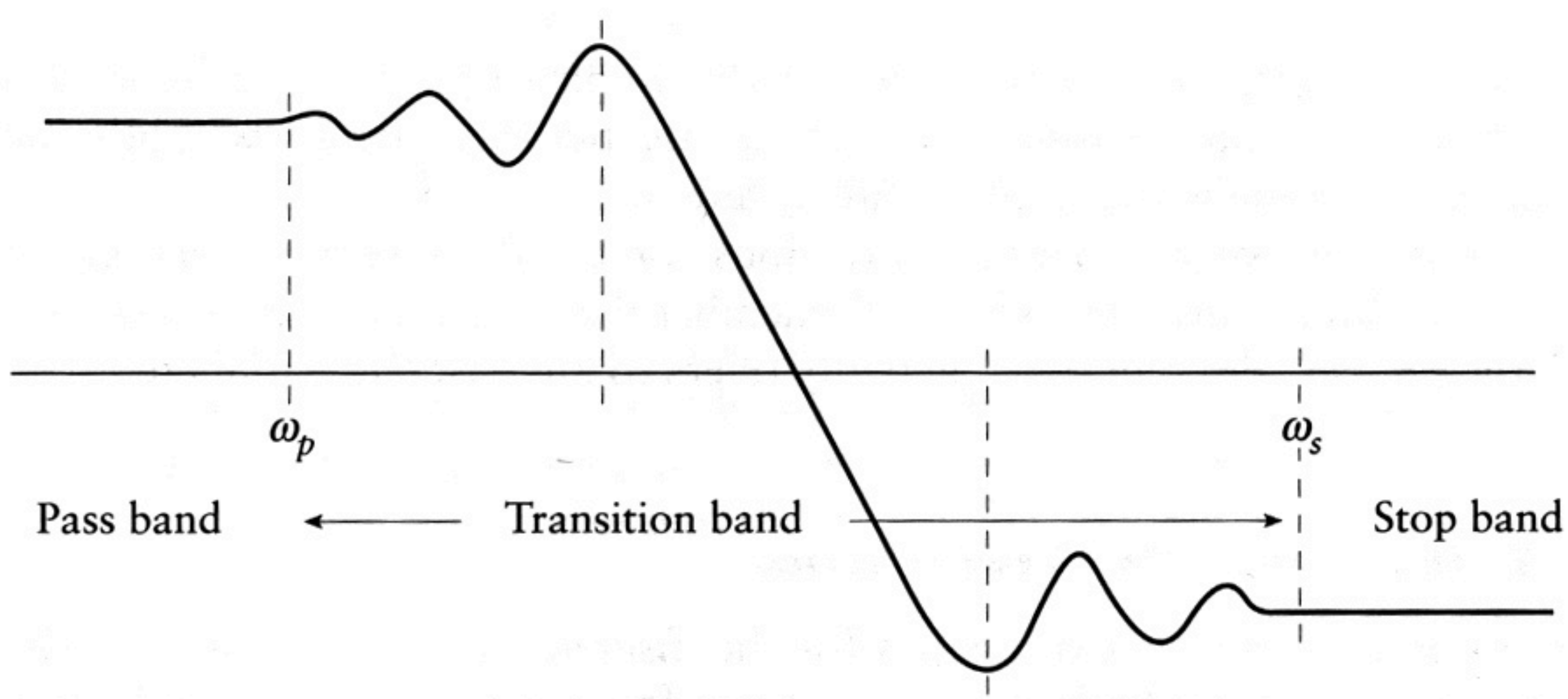


Reconstruction filters



Sampling filters

- "Ideal" is box filter in frequency
  - which is sinc function in space
- Finite support is desirable
  - compromises are necessary
- Filter design: passband, stopband, and in between



[G

Useful sampling filters

- Sampling theory gives criteria for choosing
- Box filter
  - sampling: unweighted area average
  - reconstruction: e.g. LCD
- Gaussian filter
  - sampling: gaussian-weighted area average
  - reconstruction: e.g. CRT
- Piecewise cubic
  - good small-support reconstruction filter
  - popular choice for high-quality resampling (next lecture)

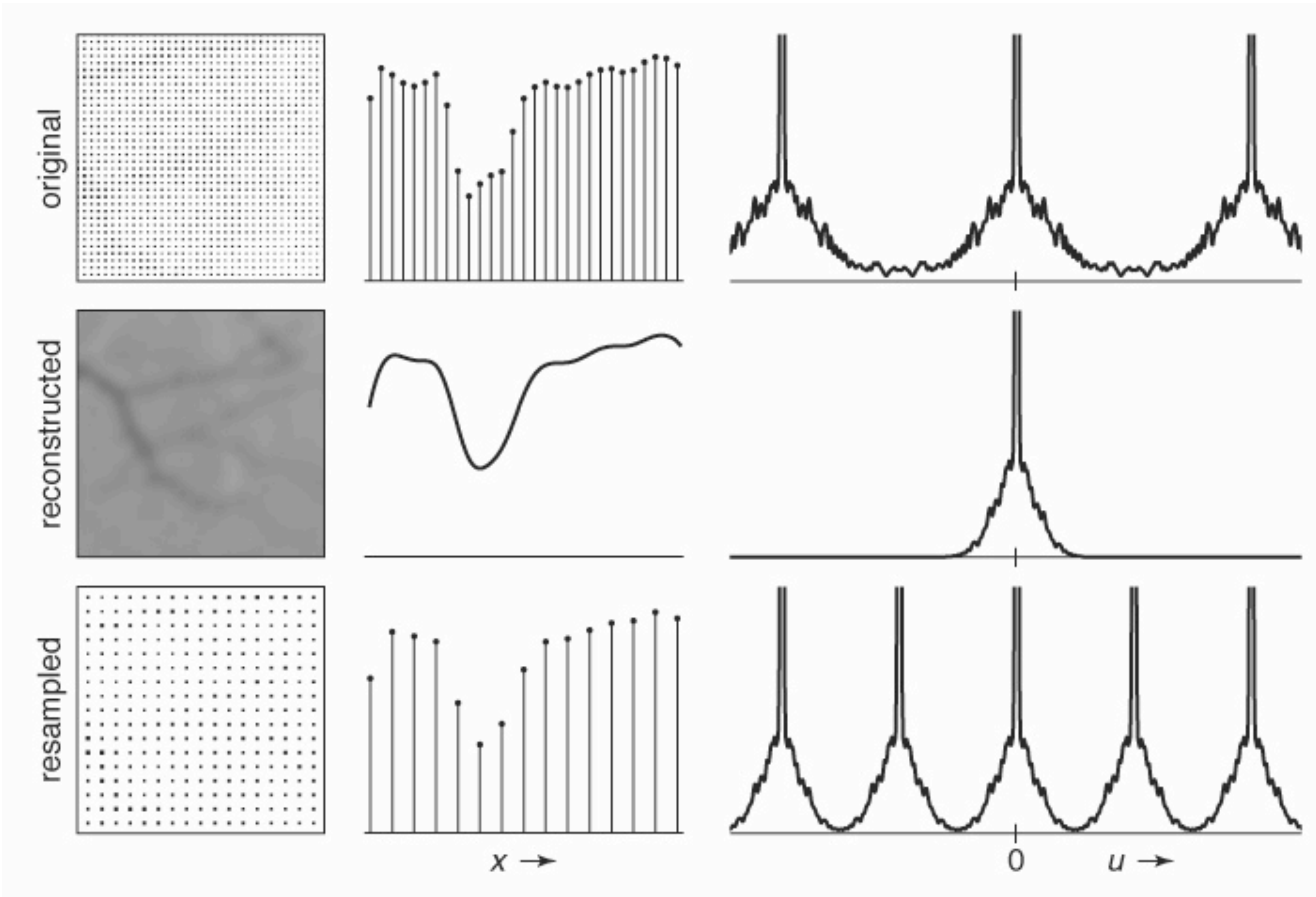
Resampling filters

- Resampling, logically, is two steps
  - first: reconstruct continuous signal
  - second: sample signal at the new sample rate
- Each step requires filtering
  - reconstruction filter
  - sampling filter
- This amounts to two successive convolutions
  - so regroup into one operation:

$$f_{\mathrm{samp}} \star f_{\mathrm{recon}} \star g = (f_{\mathrm{samp}} \star f_{\mathrm{recon}}) \star g$$

  - single filter both reconstructs and antialiases

Resampling in frequency space



Sizing reconstruction filters

- Has to perform as a reconstruction filter
  - has to be at least big enough relative to input grid
- Has to perform as a sampling filter
  - has to be at least big enough relative to output grid
- Result: filter size is max of two grid spacings
  - upsampling (enlargement): determined by input
  - downsampling (reduction): determined by output
  - for intuition think of extreme case (10x larger or smaller)

How this plays out in n-D

- Fourier transform is in terms of "plane waves"

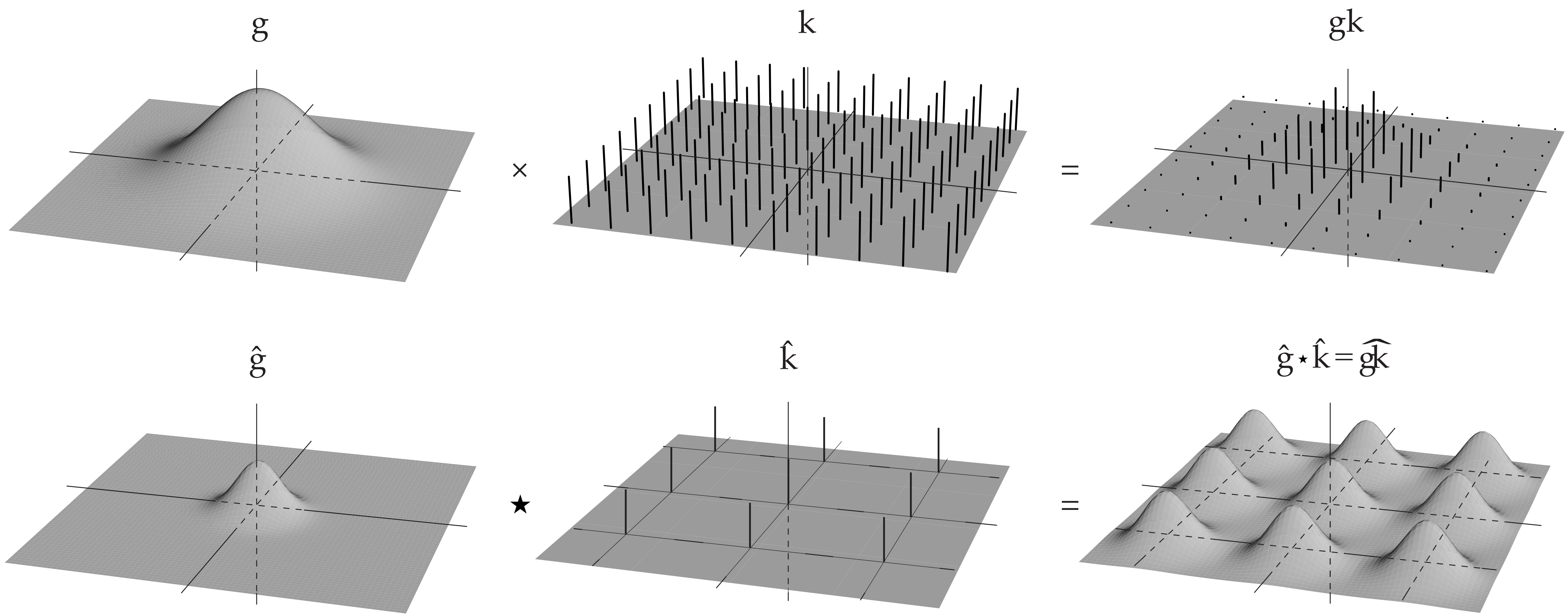
$$F(\mathbf{u}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\mathbf{u}} d\mathbf{x}$$

- Separable products of 1D functions transform separably

$$\mathcal{F}\{f(x)g(y)\} = F(u)G(v)$$

How this plays out in n-D

- By separability everything goes through the same as in 1D
  - impulse grid, filter, reconstruction
- Possibility of non-rectangular band-limiting
  - any region that does not overlap is fair game



Sampling in n-D

- With sampling on a regular lattice and reconstruction with a separable filter, everything is pretty much the same
- Non-rectangular grids are possible
  - Hexagonal arrays in 2D
  - FCC and BCC grids for volume data
  - Interlaced video
- Band limiting now means an n-D region
  - cubes are fine
  - anything that is non-overlapping is also fine

Summary

- Want to explain aliasing and answer questions about how to avoid it
- Formalized sampling and reconstruction using impulse grids and convolution
- Fourier transform gives insight into what happens when we sample
- Nyquist criterion tells us what kind of filters to use