CS6640 Computational Photography

8. Gaussian optics

First order optics

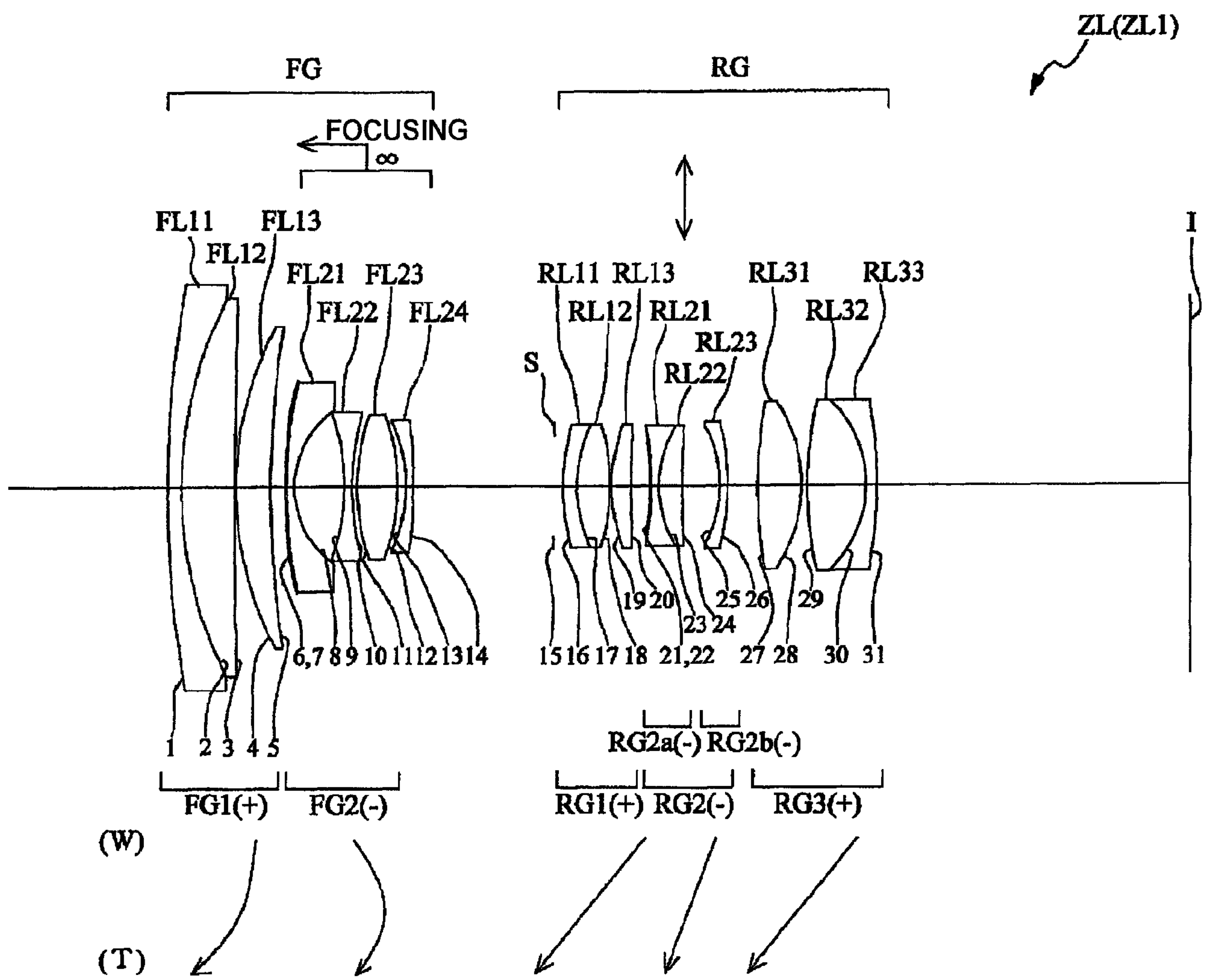
- **Lenses are complicated**

  it's all about correcting aberrations

- **If we're not interested in aberrations, it's all very simple**

  as long as you write it down using the right math

Yamaguchi (Nikon Corp.). US Patent 7,974,002 (2009)



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The thick and thin

- **Thin lenses**

  the model everyone learns first (we saw it already)

  great for building basic intuition, but missing some things that are important in practice

  lens described by just the focal length

- **Thick lenses**

  still a linear model (we'll see what this means), still simple at its core

  drops some assumptions

  models effects relevant to building cameras
  (though not so often to photographers using them)

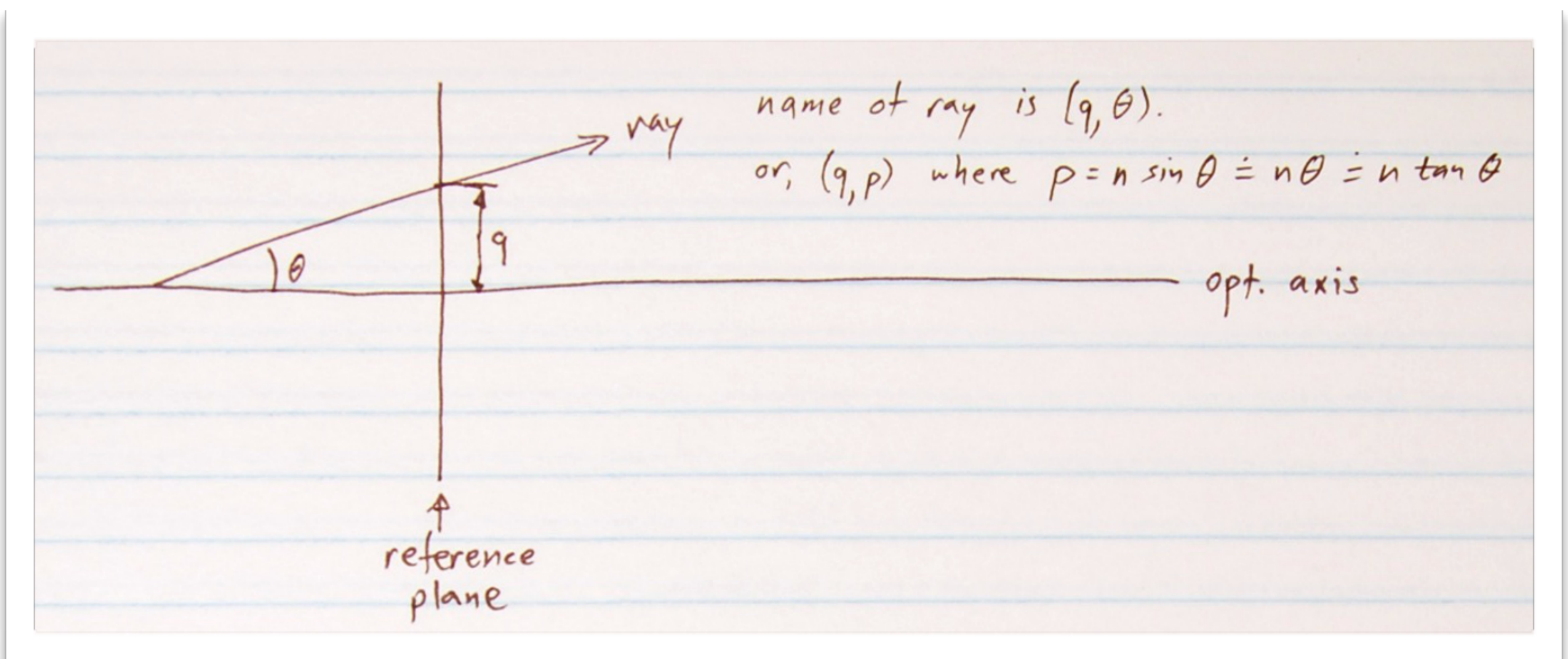
  lots more terms: principal points, nodal points, entrance/exit pupils, front/back focus distance, …

- **Watch out when reading on the web**

  people are confused about this stuff

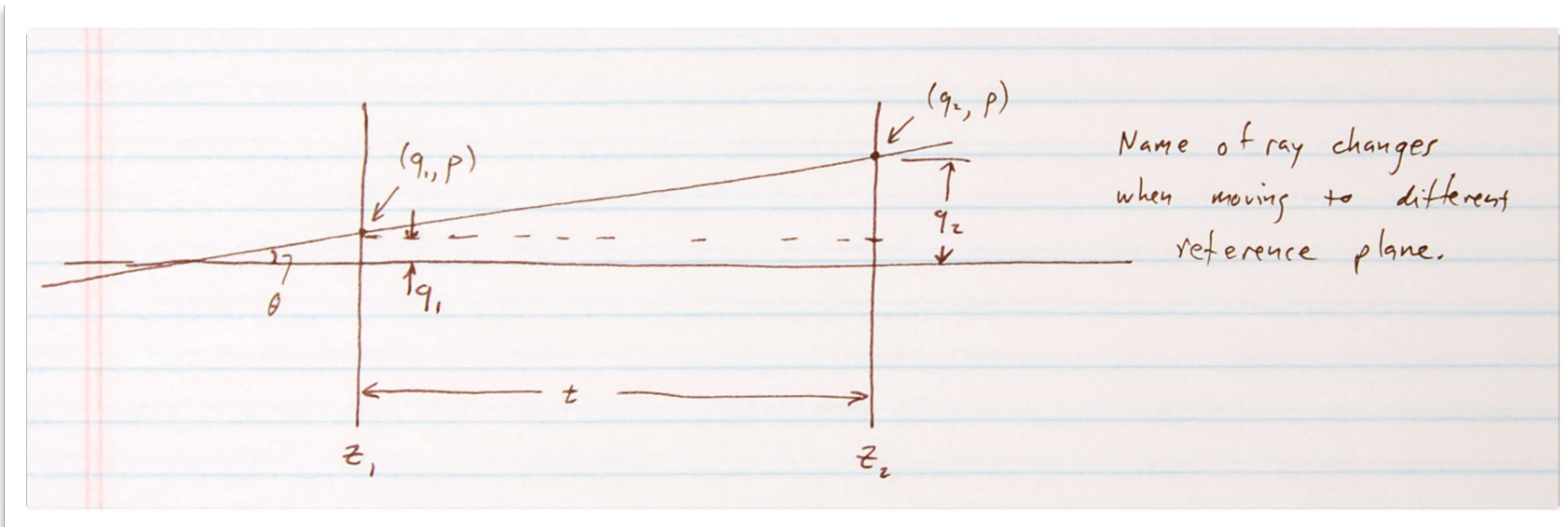
Gaussian optics

- **Optical system maps rays to rays**
  lenses are normally rotationally symmetric about an *optical axis*
- **Basic version is a first-order approximation**
  keep track just of the first derivative about the optical axis
  also assume rotational symmetry
  think only about rays coplanar with the axis
- **Under these assumptions, we just need two coordinates**



Gaussian optics

- **To first order,** θ **and** $\sin\theta$ **are interchangeable**
- **So, use** $p = n \sin\theta$ **to specify ray direction**
  the reason for including n (refractive index) comes later
- **Behavior of any optical system is a mapping from** (q, p) **to** (q, p)
- **Simple system first: empty space**



Matrix notation

- **From previous slide**

$$p_2 = p_1$$
$$q_2 = q_1 + t \tan\theta$$
$$= q_1 + Tp \quad T = t/n$$

T is called “reduced length”

- **This is a linear relationship, package it in a matrix**

$$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

$$\mathbf{T}(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

- **This gives us the name of our ray at any other reference plane**

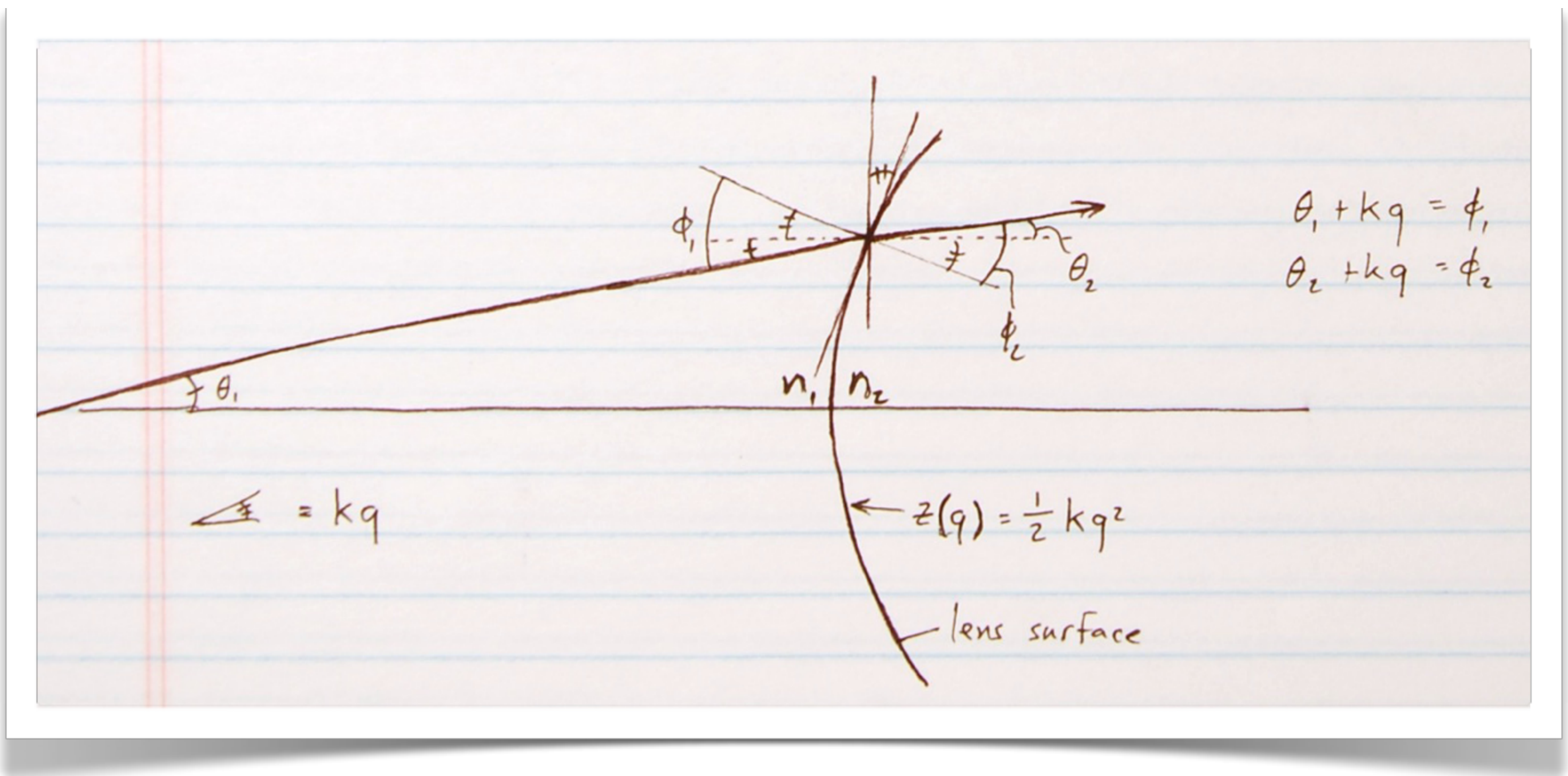
  or, it describes the effect of propagation through empty space

  p is preserved; *q* changes by an amount proportional to *p*

Refraction

- **Ray changes direction at a glass/air or glass/glass interface**
- **As usual, only need a first order model**

  this means only the curvature at the axis is relevant



Snell's law:	from diagram:	substitute:
$n_1 \sin\phi_1 = n_2 \sin\phi_2$	$\theta_i + kq = \phi_i$	$p_1 + n_1 kq = p_2 + n_2 kq$
$n_1\phi_1 \doteq n_2\phi_2$	$p_i + n_i kq = n_i\phi_i$	$p_2 = p_1 - (n_2 - n_1)kq$

Refraction — matrix

- **Again a linear relationship**

  horizontal deviation due to curvature is zero to first order

$$q_2 = q_1$$

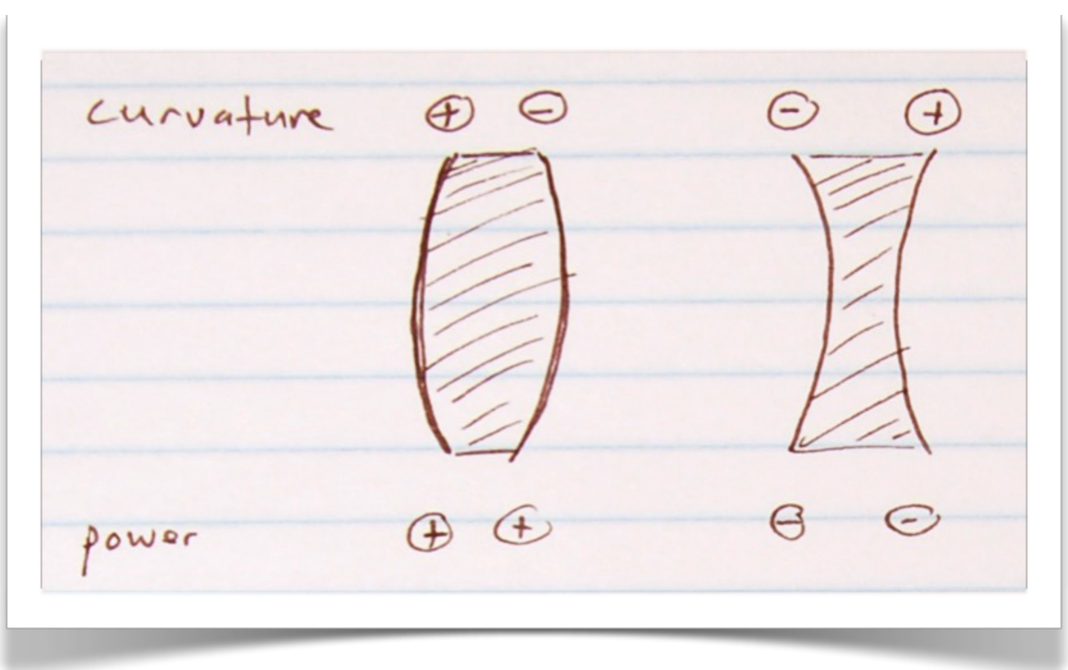
$$p_2 = p_1 - (n_2 - n_1)kq$$ *q* stays fixed; change in *p* proportional to *q*

$$= p_1 - Pq \quad P = k(n_2 - n_1)$$ *P* is the *power* of the surface

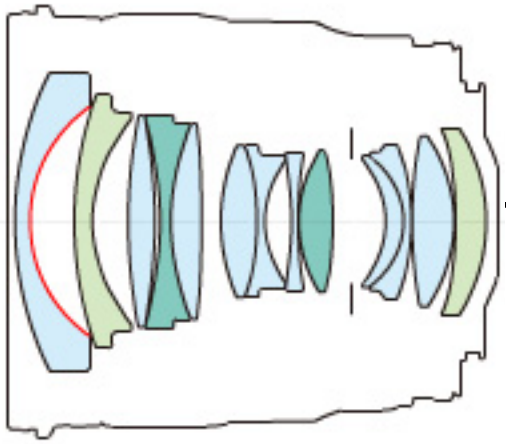
$$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix} \qquad \mathbf{R}(P) = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$$

Curvature k is positive for concave-right surfaces
Power P is the same sign as k when $n_2 > n_1$

often the radius of curvature R is provided instead of k (but same sign)



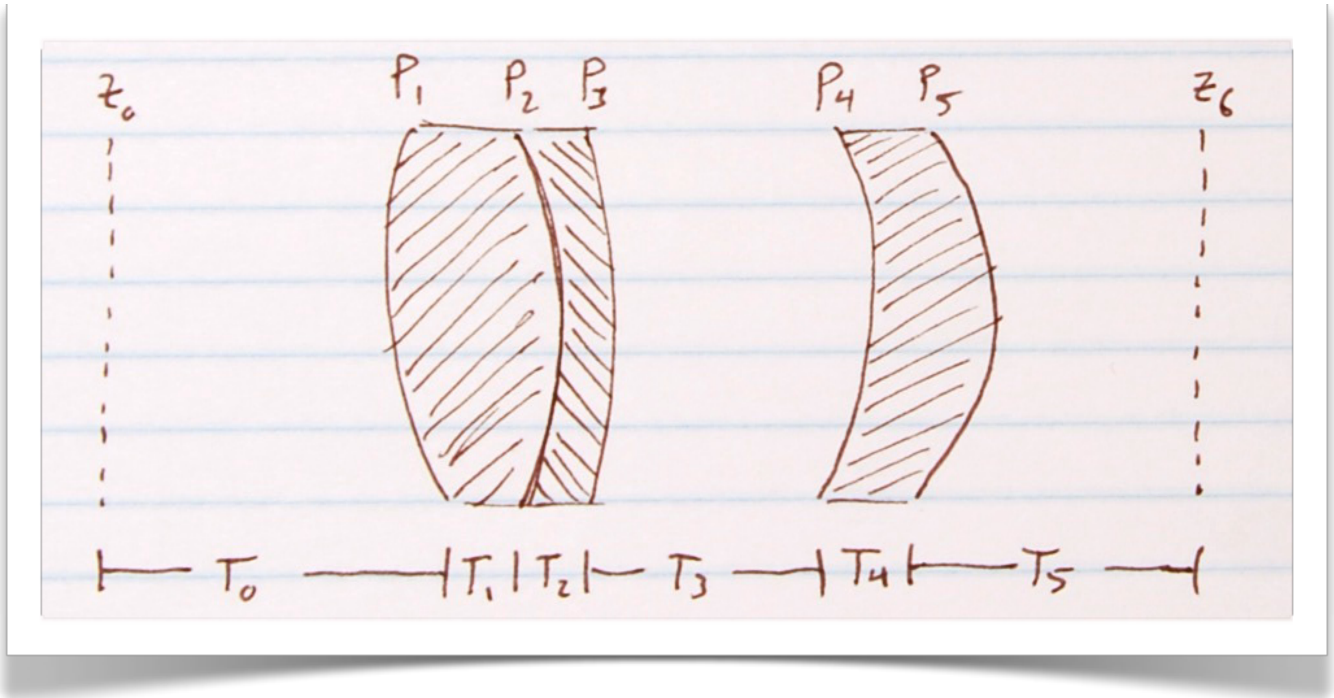
Lens systems



Canon 24mm f/1.4

- **We now have all the theory we need to handle any lens system**

  just multiply together matrices for sequence of surfaces and spaces



e.g. to find the mapping from plane z_0 to plane z_6:

$$\mathbf{M}_{0,6} = \mathbf{T}(T_5)\mathbf{R}(P_5)\mathbf{T}(T_4)\mathbf{R}(P_4)\mathbf{T}(T_3)\mathbf{R}(P_3)\mathbf{T}(T_2)\mathbf{R}(P_2)\mathbf{T}(T_1)\mathbf{R}(P_1)\mathbf{T}(T_0)$$

It’s just some 2x2 matrix!

Unit determinant

- **Not just any matrix can be formed by this kind of product**
- **Note all the R and T matrices have unit determinant**
- **Therefore all products have unit determinant**

  this means they preserve “area”

  corresponding optical concept is étendue:
  product of area and solid angle
- **So the set of possible optical systems is isomorphic to** *SL(2,R)***!**

Reading off the matrix

- **Looking at the 2x2 matrix describing a system, we can recognize some particular cases**

A = 0 $$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

This means RP2 is a focal plane q2 depends only on p1; hence parallel rays are focused.

B = 0 $$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

This means the RPs are *conjugate*: points on RP1 are imaged at points on RP2

C = 0 $$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

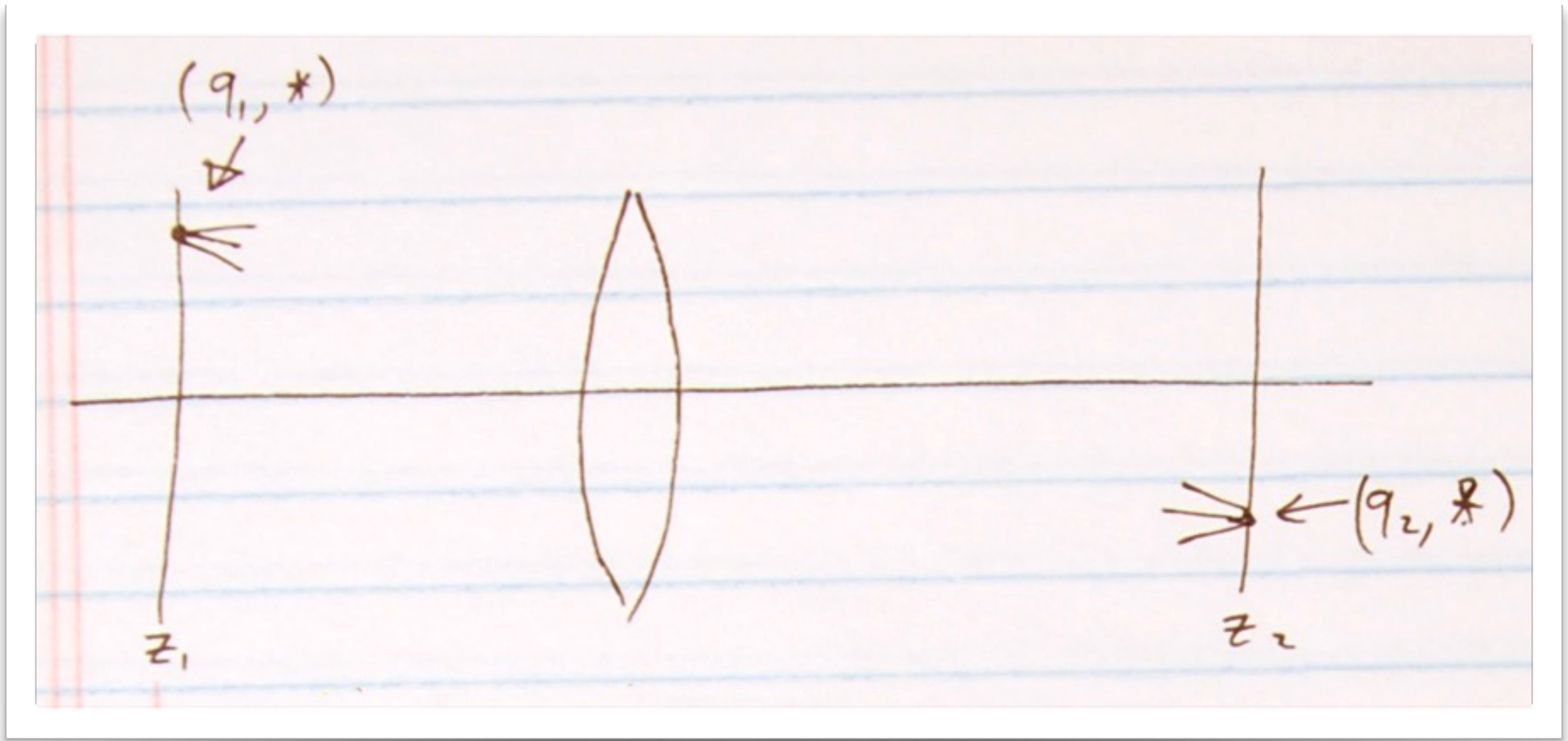
This means we have an *afocal* system (more on a later slide)

D = 0 $$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

This means RP1 is a focal plane p2 depends only on q1; hence rays from a point are collimated.

Conjugate planes

- **Two RPs for which the matrix between them has** $B = 0$
  The lens images point q1 on plane z1 at point q2 on plane z2 if all the rays through q1 map to rays that go through q2.
  In other words, q2 does not depend on p1, or B = 0.
- **Magnification can be read off the** ***A*** **entry**
- **Angular magnification (scaled by refractive index) is in** ***D***



Thin lens formula

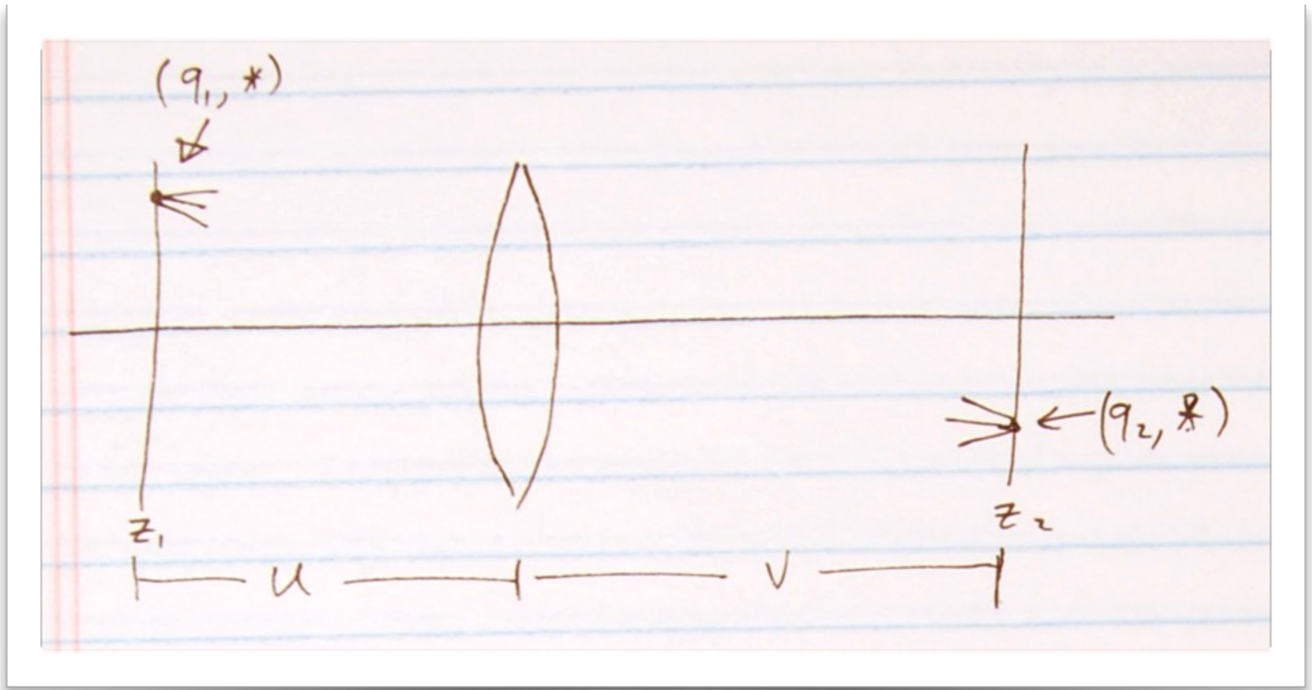
- **A lens with two refracting surfaces: what is its power?**

  if the surfaces are very close together, the powers simply add

$$\mathbf{R}(P_2)\mathbf{R}(P_1) = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(P_1 + P_2) & 1 \end{bmatrix}$$

- **Where does it produce a focused image?**

  that is, for what distances are the two RPs conjugate to one another?



$$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} \begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1 - VP & V - VUP + U \\ -P & 1 - UP \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

Thin lens formula

$$\begin{bmatrix} q_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix} \begin{bmatrix} 1 & U \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1 - VP & V - VUP + U \\ -P & 1 - UP \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \end{bmatrix}$$

- **Image forms when q_2 does not depend on p_1**

  that is, the (1,2) entry of the matrix is 0

  when this is true we say the two reference planes are *conjugate*

$$U + V - VUP = 0$$

$$\boxed{\frac{1}{U} + \frac{1}{V} = P = \frac{1}{f}}$$

result is the familiar thin lens equation!

- **Magnification is** 1 – *VP*, **which is** –(*V*/*U*)

Decomposing arbitrary matrices

- **Suppose we have an optical system characterized by some matrix. How do we interpret it?**

  factor it into the same product we used for the thin lens

$$\begin{bmatrix}1 & V\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-P & 1\end{bmatrix}\begin{bmatrix}1 & U\\0 & 1\end{bmatrix}=\begin{bmatrix}1-VP & V-VUP+U\\-P & 1-UP\end{bmatrix}=\begin{bmatrix}A & B\\C & D\end{bmatrix}$$

$$P=-C$$

$$1+VC=A \implies V=\frac{A-1}{C}$$

$$1+UC=D \implies U=\frac{D-1}{C}$$

thus for any unit-determinant 2x2 matrix,

$$\begin{bmatrix}A & B\\C & D\end{bmatrix}=\begin{bmatrix}1 & \frac{A-1}{C}\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\C & 1\end{bmatrix}\begin{bmatrix}1 & \frac{D-1}{C}\\0 & 1\end{bmatrix}=\mathbf{T}(\tfrac{A-1}{C})\mathbf{R}(-C)\mathbf{T}(\tfrac{D-1}{C})$$

Lensmaker's formula

- **Given a single-element lens with spherical surfaces**

  what is its focal length?

$$P_1 = \frac{(n-1)}{R_1} \qquad P_2 = \frac{(1-n)}{R_2} \qquad T = \frac{t}{n}$$

Convert to refractive power and reduced distance (for cleaner notation)

$$M = \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 - TP_1 & T \\ -(P_1 + P_2 - TP_1P_2) & 1 - TP_2 \end{bmatrix}$$

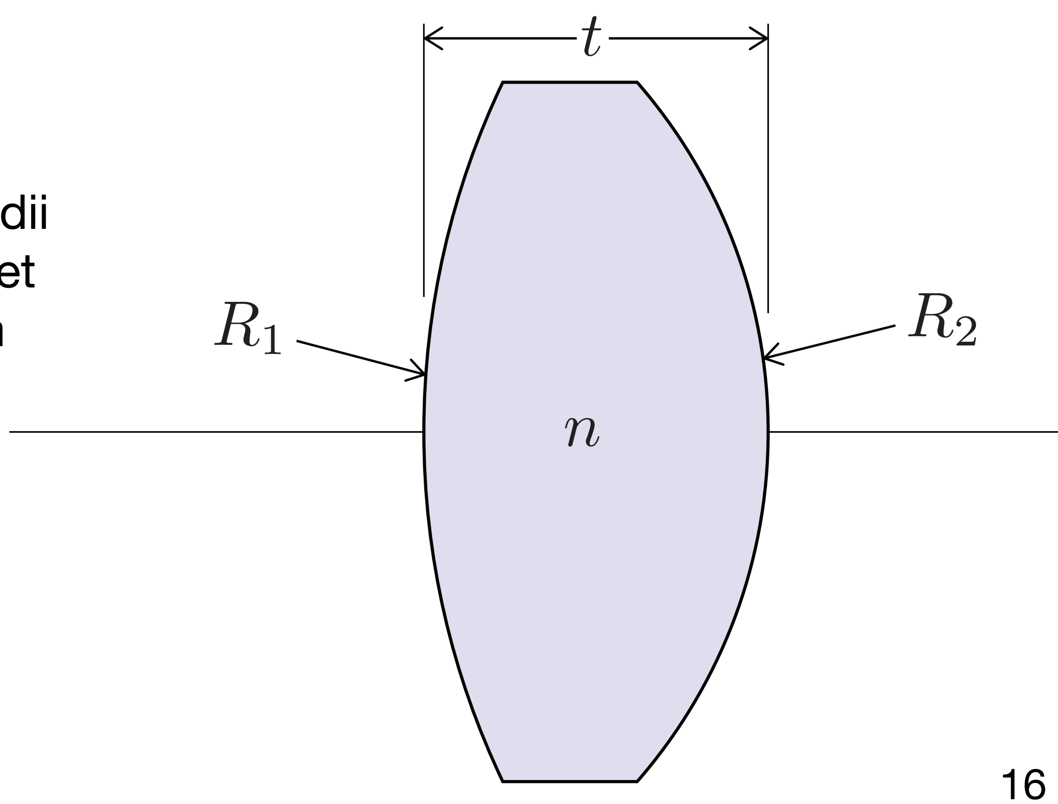
Write matrix representing this lens using reference planes at the surfaces

$$P = P_1 + P_2 - TP_1P_2$$

Read off power from the (2,1) entry

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{t(n-1)}{nR_1R_2}\right)$$

Convert back to radii and thickness to get the traditional form

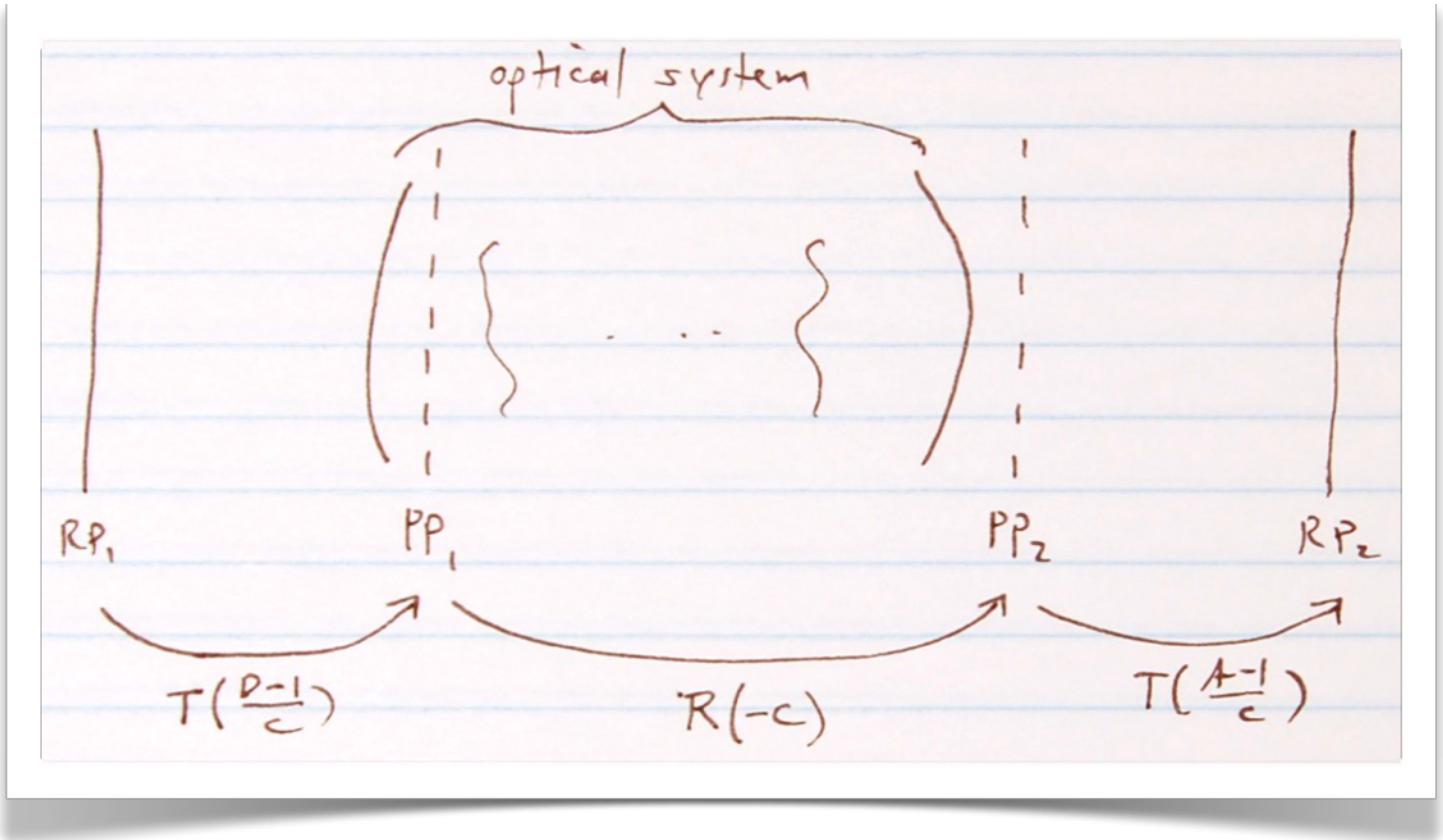


Thick lens model

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \frac{A-1}{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{D-1}{C} \\ 0 & 1 \end{bmatrix} = \mathbf{T}(\tfrac{A-1}{C})\mathbf{R}(-C)\mathbf{T}(\tfrac{D-1}{C})$$

- **This means any optical system behaves like a thin lens, but possibly with a "gap"**

  $(A - 1)/C + (D - 1)/C$ may not be the distance between the ref. planes



- **These intermediate planes, where the lens acts like a thin lens, are *principal planes***

Principal planes

- **A lens system has two principal planes (front, back or 1st, 2nd)**
- **Principal planes are where you measure from when applying the thin lens formula with a real lens**
  - in particular, magnification is the ratio of distances *to the principal planes*
- **Formal definition:**
  - the two principal planes are a pair of conjugate planes
  - the (lateral) magnification between them is +1
- **To first order a lens system has 3 degrees of freedom**
  - first principal plane (i.e. distance along axis to the 1st PP)
  - second principal plane
  - effective focal length

Gauss's cardinal points

- **Focal points**

  parallel rays focus here

- **Principal points**

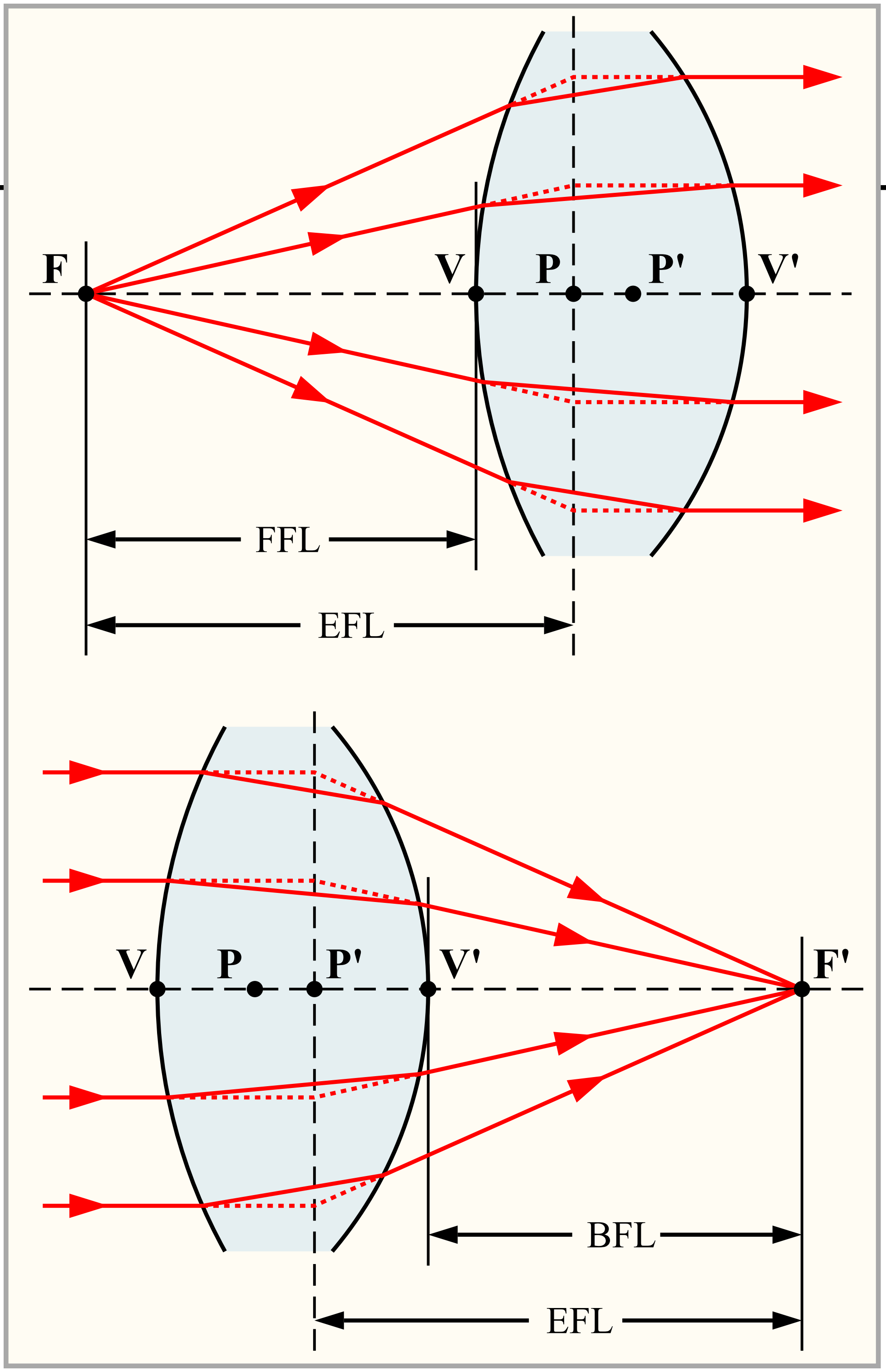
  points defining planes with unit magnification

  these are where you measure from when you want to apply the thin-lens formula

- **Nodal points**

  points defining planes with unit angular magnification

  these are the same as principal points when both sides are in air



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Other terms you may encounter: *effective focal length* (EFL): inverse of the power of the whole system; *vertex*: intersection of lens surface with axis; *back (front) focal length* (BFL, FFL): distance from focal point to vertex