CS667 Lecture Notes: Diffusion Approximation

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These notes are an excerpt from an unpublished research report by Wenzel Jakob, and they closely follow the style of presentation I used in class. This is the derivation behind the dipole model from Farrell et al. [1] that is used by Jensen et al. [3].

1 Definitions and Fundamentals

In the following section, the integration of a vertor term is to be read as a vector of integrals.

Definition 1.1. The n-th *moment* of f on the unit sphere is defined as:

$$(\mu_n[f])_{i,j,k,\ldots} := \int_{S^2} \underbrace{\omega_i \omega_j \omega_k \cdots}_{n \text{ factors}} f(\omega)\,\mathrm{d}\omega.$$

where $f : S^2 \to \mathbb{R}$.

Lemma 1.2. *Integrals of the form* $\int_0^{2\pi} \cos^n(\varphi)\sin^m(\varphi)\,d\varphi$ *are zero when* $m+n$ *is odd. This will introduce sparsity into higher-order moments of functions that are independent of the azimuth.*

Proof. Let $m, n \in \mathbb{N}$:

$$\begin{aligned}\int_0^{2\pi} \cos^{2m+1}(\varphi)\sin^{2n}(\varphi)\,\mathrm{d}\varphi &= \int_0^{2\pi} \cos^{2m}(\varphi)\cos(\varphi)\sin^{2n}(\varphi)\,\mathrm{d}\varphi \\ &= \int_0^{2\pi} (1-\sin^2(\varphi))^m \cos(\varphi)\sin^{2n}(\varphi)\,\mathrm{d}\varphi \\ &= \int_0^0 (1-x^2)^m x^{2n}\,\mathrm{d}x = 0.\end{aligned}$$

The other case is analogous. □

Corollary 1.3. *For any* f *that is independent of the azimuth when expressed in spherical coordinates:* $\int_{S^2} \omega_i \omega_j f(\omega)\,d\omega = 0$ $(i \neq j)$.

Lemma 1.4. *The 1*st *moment of a constant-valued function* f *is zero.*

Proof. Suppose that $f \equiv C \in \mathbb{R}$:

$$\begin{aligned}
\int_{S^2} \omega f(\omega)\,\mathrm{d}\omega &= \int_0^{2\pi}\int_0^{\pi} C \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= C\int_0^{2\pi} \begin{pmatrix} \frac{\pi}{2}\cos\varphi \\ \frac{\pi}{2}\sin\varphi \\ 0 \end{pmatrix}\,\mathrm{d}\varphi \\
&= 0.
\end{aligned}$$

□

Lemma 1.5. *The 0^{th} moment of a linear functional ($f(\omega) = a\cdot\omega$) is zero.*

Proof.

$$\begin{aligned}
\int_{S^2} a\cdot\omega\,\mathrm{d}\omega &= \int_{S^2} a_x\omega_1 + a_y\omega_2 + a_z\omega_3 \\
&= a_x\int_{S^2}\omega_1\,\mathrm{d}\omega + a_y\int_{S^2}\omega_2\,\mathrm{d}\omega + a_z\int_{S^2}\omega_3\,\mathrm{d}\omega \\
&= 0.
\end{aligned}$$

□

Lemma 1.6. *The 1^{st} moment of a linear functional ($f(\omega) = a\cdot\omega$) is $\dfrac{4\pi}{3}\,a$.*

Proof. The components of $\int_{S^2}\omega\,(a\cdot\omega)\,\mathrm{d}\omega$ are:

$$\int_{S^2} w_i\,(a\cdot\omega)\,\mathrm{d}\omega = \sum_{j=1}^{3} a_j\int_{S^2}\omega_i\,\omega_j\,\mathrm{d}\omega$$

Since $\int_0^{\pi}\sin^2\theta\cos\theta\,\mathrm{d}\theta = 0$ and $\int_0^{2\pi}\sin\varphi\cos\varphi\,\mathrm{d}\varphi = 0$, the summands with $i \neq j$ will all be zero. Furthermore, because in spherical coordinates $\|\omega\|$ is equal to one,

$$4\pi = \int_{S^2} 1\,\mathrm{d}\omega = \int_{S^2}\omega_1^2\,\mathrm{d}\omega + \int_{S^2}\omega_2^2\,\mathrm{d}\omega + \int_{S^2}\omega_3^2\,\mathrm{d}\omega.$$

For reasons of symmetry, the summands have identical values:

$$\int_{S^2} w_i^2\,\mathrm{d}\omega = \frac{4\pi}{3}\ \ (i = 1,2,3)$$

and thus

$$\int_{S^2}\omega\,(a\cdot\omega)\,\mathrm{d}\omega = \frac{4\pi}{3}\,a.$$

□

Lemma 1.7. *The 0^{th} moment of a quadratic form $f(\omega) = \omega^T A\omega$ is $\dfrac{4\pi}{3}\,\mathrm{Tr}(A)$.*

Proof. Similarly to before, symmetry causes most summands to vanish:

$$\begin{aligned}\int_{S^2} \omega^T \cdot A\omega \, \mathrm{d}\omega &= \int_{S^2} \sum_{i=1}^{n} \omega_i \sum_{j=1}^{n} a_{ij}\omega_j \, \mathrm{d}\omega \\ &= \sum_{i=1}^{n} a_{ii} \int_{S^2} \omega_i^2 \, \mathrm{d}\omega \\ &= \frac{4\pi}{3} \operatorname{Tr}(A)\end{aligned}$$

□

Lemma 1.8. *The* 1^{st} *moment of a quadratic form* $f(\omega) = \omega^T A\omega$ *is zero.*

Proof. The components of $\int_{S^2} (\omega^T \cdot A\omega)\omega \, \mathrm{d}\omega$ are:

$$\int_{S^2} \omega_i \sum_{k=1}^{n} \sum_{l=1}^{n} \omega_k a_{kl} \omega_l \, \mathrm{d}\omega = a_{ii} \int_{S^2} \omega_i^3 \, \mathrm{d}\omega = 0.$$

□

Definition 1.9. (i) *Radiant fluence* is defined as the 0^{th} moment of L while keeping x fixed:

$$\phi(x) = \mu_0[L(x, \cdot)] = \int_{S^2} L(x, \omega) \, \mathrm{d}\omega$$

(ii) *Vector irradiance* is defined as the 1^{st} moment of L while keeping x fixed:

$$\vec{E}(x) = \mu_1[L(x, \cdot)] = \int_{S^2} \omega L(x, \omega) \, \mathrm{d}\omega$$

(iii) The *isotropic phase function* is defined as a function $\rho : S^2 \times S^2 \to \mathbb{R}$, which additionally satisfies the constraints

(a) $\int_{S^2} \rho(\omega, \omega') \, \mathrm{d}\omega' = 1 \; \forall \omega \in S^2$ (probability distribution)

(b) $\rho(\omega, \omega') = \rho(\omega \cdot \omega')$ (rotational symmetry)

(iv) $g \in \mathbb{R}$ is defined as the averaged forward scattering minus the backward scattering of ρ [2]:

$$g := \int_{S^2} \rho(\omega, \omega') \, \omega \cdot \omega' \, \mathrm{d}\omega'$$

Lemma 1.10. *When fixing the incident direction of* ρ *to* $\omega_0 \in S^2$*, its moments are:*

(i) $\mu_0[\rho(\omega_0, \cdot)] = 1$.

(ii) $\mu_1[\rho(\omega_0, \cdot)] = g\omega_0$ $(\omega_0 \in S^2)$.

Proof. (i) Follows directly from definition 1.9 (iii).

(ii) The vector ω_0 can be extended to an ONB (ω_0, u, v) of R^3. Calculating μ_1 in this space with w_0 as a fixed argument of ρ yields the first component

$$\int_{S^2} \rho(\omega_0, \omega') \, (\omega_0 \cdot \omega') \, \mathrm{d}\omega' = g. \quad \text{(Definition 1.9 (iv))}$$

Partitioning the unit sphere into two hemispheres around u and $-u$ will cause the two integral summands to have opposite signs due to the rotational symmetry:

$$\int_{S^2} \rho(\omega_0, \omega')\,(u \cdot \omega')\,\mathrm{d}\omega' = \int_{H^2(u)} \rho(\omega_0, \omega')\,(u \cdot \omega')\,\mathrm{d}\omega' + \int_{H^2(-u)} \rho(\omega_0, \omega')\,(u \cdot \omega')\,\mathrm{d}\omega' = 0.$$

Analogous holds for v. Thus, $\mu_1[\rho(\omega_0, \cdot)] = g\omega_0$.

$\square$

2 Derivation of the Isotropic Diffusion Equation

We make the assumption that L is directionally smooth and well-approximated by a first-order expansion:

$$L(x, \omega) := \frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega \cdot \vec{E}(x).$$

The *radiative transfer equation* (RTE) is given by

$$(\omega \cdot \nabla)L(x, \omega) + \sigma_t L(x, \omega) = \sigma_s \int_{S^2} \rho(x, \omega, \omega')L(x, \omega')\,\mathrm{d}\omega' + Q(x, \omega). \tag{1}$$

Substitution of the first-order expansion of L into the RTE results in:

$$\begin{aligned}(\omega \cdot \nabla)\left(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega \cdot \vec{E}(x)\right) + \sigma_t\left(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega \cdot \vec{E}(x)\right) = \\ \sigma_s \int_{S^2} \rho(x, \omega, \omega')\left(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega' \cdot \vec{E}(x)\right)\mathrm{d}\omega' + Q(x, \omega).\end{aligned} \tag{2}$$

With the restricted representation of L, we can no longer expect to be able to solve the RTE exactly. Instead, we will project both sides of the equation into this reduced space and search for equality amongst the 0^{th} and 1^{st}-order coefficients.

Left hand side (0^{th} order)

$$\begin{aligned}&\mu_0\left[(\omega \cdot \nabla)\left(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega \cdot \vec{E}(x)\right) + \sigma_t\left(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega \cdot \vec{E}(x)\right)\right] \\ &= \frac{1}{4\pi}\underbrace{\int_{S^2} \omega \cdot \nabla\phi(x)\,\mathrm{d}\omega}_{=0\ (1.5)} + \frac{3}{4\pi}\underbrace{\int_{S^2} \omega \cdot \nabla(\vec{E}(x) \cdot \omega)\,\mathrm{d}\omega}_{=\frac{4\pi}{3}\,\mathrm{Tr}(\nabla\vec{E}) = \frac{4\pi}{3}\,\mathrm{div}\,\vec{E}\ (1.7)} + \frac{\sigma_t}{4\pi}\underbrace{\int_{S^2} \phi(x)\,\mathrm{d}\omega}_{=4\pi\phi(x)} + \frac{3\,\sigma_t}{4\pi}\underbrace{\int_{S^2} \omega \cdot \vec{E}(x)\,\mathrm{d}\omega}_{=0\ (1.5)} \\ &= \mathrm{div}\,\vec{E}(x) + \sigma_t\,\phi(x).\end{aligned} \tag{3}$$

Right hand side (0^{th} order)

$$\begin{aligned}&\mu_0\left[\sigma_s \int_{S^2} \rho(x, \omega, \omega')\left(\frac{1}{4\pi}\phi(x) + \frac{3}{4\pi}\,\omega' \cdot \vec{E}(x)\right)\mathrm{d}\omega' + Q(x, \omega)\right] \\ &= \frac{\sigma_s}{4\pi}\underbrace{\int_{S^2}\int_{S^2} \rho(x, \omega, \omega')\phi(x)\,\mathrm{d}\omega\,\mathrm{d}\omega'}_{=4\pi\phi(x)\ (\phi\ \text{const},\ (1.10\ \text{i}))} + \frac{3\,\sigma_s}{4\pi}\underbrace{\int_{S^2}\int_{S^2} \rho(x, \omega, \omega')\,\omega' \cdot \vec{E}(x)\,\mathrm{d}\omega\,\mathrm{d}\omega'}_{=0\ (E\ \text{const},\ (1.10\ \text{ii}),\ (1.5))} + Q_0(x) \\ &= \sigma_s\phi(x) + Q_0(x)\end{aligned} \tag{4}$$

where $Q_0(x) := \mu_0[Q(x, \cdot)]$.

Resulting equation

The resulting equation intuitively expresses that the divergence of the vector irradiance field $\vec{E}$ is positive in the vicinity of sources ($Q_0 > 0$) and negative in the presence of absorption.

$$\begin{aligned} & \operatorname{div} \vec{E}(x) + \sigma_t\, \phi(x) = \sigma_s \phi(x) + Q_0(x) \\ \Leftrightarrow \quad & \operatorname{div} \vec{E}(x) = -\sigma_a \phi(x) + Q_0(x) \end{aligned} \tag{5}$$

Left hand side (1st order)

$$\begin{aligned} & \mu_1 \left[(\omega \cdot \nabla) \left(\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi}\, \omega \cdot \vec{E}(x) \right) + \sigma_t \left(\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi}\, \omega \cdot \vec{E}(x) \right) \right] \\ & = \frac{1}{4\pi} \underbrace{\int_{S^2} \omega (\omega \cdot \nabla \phi(x))\, \mathrm{d}\omega}_{= \frac{4\pi}{3} \nabla \phi(x) \ (1.6)} + \frac{3}{4\pi} \underbrace{\int_{S^2} \omega (\omega^T \nabla \vec{E}(x)\, \omega)\, \mathrm{d}\omega}_{= 0 \ (1.8)} + \frac{\sigma_t}{4\pi} \underbrace{\int_{S^2} \omega \phi(x)\, \mathrm{d}\omega}_{= 0 \ (1.4)} + \frac{3\, \sigma_t}{4\pi} \underbrace{\int_{S^2} \omega (\omega \cdot \vec{E}(x))\, \mathrm{d}\omega}_{= \frac{4\pi}{3} \vec{E}(x) \ (1.6)} \\ & = \frac{1}{3} \nabla \phi(x) + \sigma_t \vec{E}(x) \end{aligned} \tag{6}$$

Right hand side (1st order)

$$\begin{aligned} & \mu_1 \left[\sigma_s \int_{S^2} \rho(x, \omega, \omega') \left(\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi}\, \omega' \cdot \vec{E}(x) \right) \mathrm{d}\omega' + Q(x, \omega) \right] \\ & = \frac{\sigma_s}{4\pi} \underbrace{\int_{S^2} \int_{S^2} \omega \rho(x, \omega, \omega') \phi(x)\, \mathrm{d}\omega\, \mathrm{d}\omega'}_{= 0 \ (\phi \text{ const, } (1.10 \text{ ii}),\ (1.5))} + \frac{3\, \sigma_s}{4\pi} \underbrace{\int_{S^2} \int_{S^2} \omega \rho(x, \omega, \omega')\, \omega' \cdot \vec{E}(x)\, \mathrm{d}\omega\, \mathrm{d}\omega'}_{= g\, \frac{4\pi}{3} \vec{E}(x) \ (\text{rearrange, } (1.10 \text{ ii}),\ (1.6))} + Q_1(x) \\ & = g \sigma_s \vec{E}(x) + Q_1(x) \end{aligned} \tag{7}$$

where $Q_1(x) \coloneqq \mu_1[Q(x, \cdot)]$.

Resulting equation

The 1st -order equation can be re-written as follows:

$$\begin{aligned} & \frac{1}{3} \nabla \phi(x) + \sigma_t \vec{E}(x) = g \sigma_s \vec{E}(x) + Q_1(x) \\ \Leftrightarrow \quad & \frac{1}{3} \nabla \phi(x) = (g \sigma_s - \sigma_t) \vec{E}(x) + Q_1(x) \\ \Leftrightarrow \quad & \frac{1}{3} \nabla \phi(x) = - \underbrace{(\sigma_a + (1 - g) \sigma_s)}_{=: \sigma_{t'}} \vec{E}(x) + Q_1(x) \\ \Leftrightarrow \quad & \nabla \phi(x) = -3 \sigma_{t'} \vec{E}(x) + 3 Q_1(x) \end{aligned} \tag{8}$$

where $\sigma_{t'} = \sigma_a + \sigma_{s'}$ and $\sigma_{s'} = (1 - g) \sigma_s$ are the reduced transport and scattering coefficients, respectively.

Putting it together

Solving (8) for $\vec{E}(x)$ results in

$$\vec{E}(x) = \frac{1}{\sigma_{t'}}\, Q_1(x) - \frac{1}{3 \sigma_{t'}} \nabla \phi(x),$$

which can be substituted into (5):

$$\begin{aligned} & \operatorname{div}\left(\frac{1}{\sigma_{t'}} Q_1(x) - \frac{1}{3\sigma_{t'}} \nabla\phi(x)\right) = -\sigma_a \phi(x) + Q_0(x) \\ \Leftrightarrow \quad & \frac{1}{\sigma_{t'}} \operatorname{div} Q_1(x) - \frac{1}{3\sigma_{t'}} \nabla^2 \phi(x) = -\sigma_a \phi(x) + Q_0(x) \\ \Leftrightarrow \quad & D\nabla^2 \phi(x) = \sigma_a \phi(x) - Q_0(x) + 3D \operatorname{div} Q_1(x) \qquad (9) \end{aligned}$$

where $D = \dfrac{1}{3\sigma_{t'}}$.

References

[1] Farrell, T. J., and Patterson, M. S. A diffusion theory model of spatially resolved, steady-state diffuse reflectance for the noninvasive determination of tissue optical properties in vivo. In *Medical Physics, Volume 19, Issue 4* (1992), pp. 879–888.

[2] Ishimaru, A. *Wave Propagation and Scattering in Random Media*. Academic Press, New York, USA, 1978.

[3] Jensen, H. W., Marschner, S. R., Levoy, M., and Hanrahan, P. A practical model for subsurface light transport. In *SIGGRAPH '01: Proceedings of the 28th annual conference on Computer graphics and interactive techniques* (New York, NY, USA, 2001), ACM, pp. 511–518.