CS 624: Numerical Solution of Differential Equations
Spring 2006
Problem Set 5

Handed out: Wed., Apr. 5.

Due: Fri., Apr. 14 in lecture.

1. Consider the Lax-Friedrichs method for a scalar conservation law $u_t = -(f(u))_x$. Show that Lax-Friedrichs is actually second-order for a modified PDE $u_t = -(f(u))_x + T$, where T is a sum of some other terms that you must determine, all of which should be $O(k)$. The formula for T will involve h and k. In your analysis, assume $k/h = \lambda$, where λ is a fixed constant.

   If you carry this out correctly, T will have three terms, two of which are multiples u_{xx}. If the two latter terms are added to arrive at a single term Au_{xx}, is the coefficient A positive or negative? Take into account the CFL condition.

   [Hint: Plug in the exact solution and carry out a Taylor series expansion. Figure out how to define u_t (i.e., figure out what T has to be) in such a way that the ku_t term on the left-hand side of the Taylor expansion can cancel the $O(k^2)$ term on the left-hand side and the $O(h^2)$ term on the right-hand side.]

2. (a) Show that the viscous Burgers equation $u_t + uu_x = au_{xx}$, where $a > 0$ is a constant, can be transformed analytically to a diffusion equation.

   [Hint: Use the "Cole-Hopf transformation": Show that if v satisfies $v_t + v_x^2/2 = av_{xx}$, then $u = v_x$ is a Burgers solution. Then substitute $v = -2a \log w$, coming up with a new equation for w.]

   (b) Note that the diffusion equation $w_t = aw_{xx}$ defined on $[0, 2\pi] \times [0, \infty)$ with periodic BC's (i.e., $w(0, t) = w(2\pi, t)$ for all $t \geq 0$) has as an exact analytic solution $w(x, t) = C + \exp(-n^2 at) \sin(nx)$ for any integer n and real constant C. Figure out the corresponding Burgers' analytic solution.

3. Consider the scalar hyperbolic conservation law $u_t + (f(u))_x = 0$. Assume the flux function f is convex, i.e., f' is an increasing function. Verify that Godunov's method is an "E-scheme." (This is terminology introduced by Osher; it is a step in the proof that Godunov's method is TVD. See LeVeque's second book for more information about E-schemes.) The definition of E-scheme in the special case of Godunov's method is that the inequality $(b - a)(f(c) - f(q)) \leq 0$ must hold for any a, b and for c as defined by the lecture notes on Godunov's method and for any q between a and b (i.e., $q \in [a, b]$ or $q \in [b, a]$ depending on whether $a < b$ or not). In terms of the notation used in LeVeque's first book (*Numerical Methods for Conservation Laws*), p. 144, the E-scheme condition is that $(u_r - u_l)(f(u^*) - f(q)) \leq 0$ for any u_l and u_r and for any q between u_l and u_r.

4. Implement (in Matlab) the Lax-Friedrichs and Godunov method for Burgers' equation with Riemann initial condition

$$u_0(x) = \begin{cases} 2 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

   Use spatial interval $[-1, 1]$ and integrate up to $T = 0.2$. Try various values of k and h, ensuring that the CFL condition is satisfied. Use the obvious Dirichlet boundary conditions (2 at the left, 1 at the right).

   Develop a heuristic to measure the width of the shock. Determine experimentally how much Lax-Friedrichs smears the shock (as a function of k), and how much Godunov smears the shock.

   Hand in listings of all m-files, at least two interesting plots, and a paragraph of conclusions.