CS 624: Numerical Solution of Differential Equations
Spring 2006
Problem Set 4

Handed out: Mon., Mar. 13.

Due: Wed., Mar. 29 in lecture.

1. Find an additional term of the form

$$\theta v_{j-2}^n + \gamma v_{j-1}^n - \gamma v_{j+1}^n - \theta v_{j+2}^n$$

that may be added to Lax-Wendroff formula in order to make it 3rd order for the one-way wave equation. Analyze the stability of the method. Note that θ and γ will depend on k and h. In the stability analysis, full credit will be given for identifying a reasonable range of choices of k, h that makes the method stable even if the complete range of stable choices is not identified.

2. In their original paper, CFL considered the following two-step explicit finite difference scheme for the full wave equation $u_{tt} = u_{xx}$:

$$v_j^{n+1} = 2v_j^n - v_j^{n-1} + \lambda^2(v_{j+1}^n - 2v_j^n + v_{j-1}^n).$$

Rewrite this as an explicit one-step vector finite difference method. Then apply a Fourier transform to the vector formula (see 3.6 of the text if you don't know how to do this) to determine the amplification factor $G(\xi)$. Finally, determine upper bounds on the absolute values of the eigenvalues of $G(\xi)$ assuming $\lambda \leq 1$.

3. Design a finite difference method for the PDE

$$u_t = au + u_{xx}$$

where a is a nonpositive constant real number. This equation models heat flow in which the heat simultaneously diffuses and flows out of the object (because the object is soaking in an ice-bath, for instance). It is a special case of a class of PDE's called "reaction-diffusion equations." Show that the order of accuracy for your finite difference method is positive. Show that your method is stable. Show that your method is convergent (by citing the Lax equivalence theorem); your convergence argument should include an explicit relation among h, k, a.

4. Consider the degenerate nonlinear parabolic equation of the form $u_t = (u^2)_{xx}$ on the interval $[-1, 1]$. Solve this equation in matlab using the "method of lines" (see the first two pages of 3.3 of the text). That is, first discretize in space only: Replace the space-derivatives in the above equation with a difference approximation. Then apply `ode15s` to this system of ODEs. Use as initial conditions $u(x, 0) = \max(.25 - x^2, 0)$. Use as boundary conditions $u(-1, t) = u(1, t) = 0$.

Integrate to $t = 1$ and make a plot of $u(.75, t)$ as a function of t. Carry out a mesh refinement study to see how the plot varies as the mesh is refined. Describe in words the apparent behavior of $u(.75, t)$.

Hand in listings of your m-files, a few sentences of conclusions, and at least two interesting plots.