CS 624: Numerical Solution of Differential Equations
Spring 2006
Problem Set 3

Handed out: Mon., Feb. 27.

Due: Wed., Mar. 8 in lecture.

1. Consider the Kepler problem from PS1, which describes the motion of a light body orbiting a heavy body:
$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{\mathbf{x}}{\|\mathbf{x}\|^3}.$$
Here, $\mathbf{x}$ is a vector in $\mathbf{R}^2$. Let
$$H(\mathbf{x},\mathbf{v}) = \frac{\|\mathbf{v}\|^2}{2} - \frac{1}{\|\mathbf{x}\|}.$$
It is easy to check that the above 2nd order ODE, when converted to a first order system using the standard trick of defining $\mathbf{v} = d\mathbf{x}/dt$, is a Hamiltonian system of ODE's for the above energy function.

   (a) Show that if the leapfrog method is used to integrate the Kepler system, then
$$H(\mathbf{x}^{n+1},\mathbf{v}^{n+1}) - H(\mathbf{x}^n,\mathbf{v}^n) = k^2\left(-\frac{1}{2\|\mathbf{x}\|^4} - \frac{3(\mathbf{v}^T\mathbf{x})^2}{2\|\mathbf{x}\|^5} + \frac{\mathbf{v}^T\mathbf{v}}{2\|\mathbf{x}\|^3}\right) + O(k^3).$$

   (b) Construct a perturbed energy function $\bar{H}(\mathbf{x},\mathbf{v})$ (that may also depend on k) with the property that if the sequence $(\mathbf{x}^n,\mathbf{v}^n)$ is defined as above (i.e., as the solution computed by leapfrog to the original Kepler problem), then $\bar{H}(\mathbf{x}^{n+1},\mathbf{v}^{n+1}) - \bar{H}(\mathbf{x}^n,\mathbf{v}^n) = O(k^3)$. Your function $\bar{H}(\mathbf{x},\mathbf{v})$ should have the form
$$\bar{H}(\mathbf{x},\mathbf{v}) = H(\mathbf{x},\mathbf{v}) + kf(\mathbf{x},\mathbf{v}),$$
where $f(\mathbf{x},\mathbf{v})$ is a function that you determine explicitly. [A hint for part (b) is to first regard x and v as scalars and guess that in the scalar analog of this problem, f has the form $f(x,v) = Ax^dv^e$. Once the solution to the scalar analog is found, see if you can generalize to the case that $\mathbf{x}$ and $\mathbf{v}$ are vectors.]

2. Consider applying the leapfrog method to the partitioned Hamiltonian problem whose Hamiltonian is given by $H(\mathbf{p},\mathbf{q}) = (\mathbf{p}^TA\mathbf{p} + \mathbf{q}^TB\mathbf{q})/2$ where A, B are positive-definite diagonal matrices. Determine the condition that assures that the numerical method is stable (i.e., does not exhibit unbounded growth as $n \to \infty$). Your condition should be in as simple form as possible, involving k and the diagonal entries of A, B.

3. Symplectic integrators for Hamiltonian problems do not conserve energy, as mentioned in lecture. One way to get exact energy conservation for Hamiltonian problems is to drop one of the equations in the ODE system, and replace it with a constraint that energy is conserved. Then the resulting problem can be solved as a DAE. This method often fails.

   Solve the Kepler problem (Q1) in this manner. Use `ode15s`. Note that `ode15s` solves index-one DAE's in semi-explicit form if you give it a diagonal matrix M as input that has a mixture of 1's (evolution equations) and 0's (constraints) on the main diagonal. Try replacing one of the x_i evolution equations with the constraint, and also one of the v_i evolution equations. Hand in m-file listings, plots, and a brief statement of your conclusions. Explain why this method doesn't work very well. (Hint: consider the index.)

4. Transform the Kepler system of ODEs into polar coordinates. In particular, let the change of variables be $\mathbf{x} = r(\cos\theta, \sin\theta)$ and $\mathbf{v} = w(\cos\beta, \sin\beta)$ where r, θ, w, β are the new time-dependent variables. (Simplify as much as possible.) Argue that it is possible to replace one evolution equation in the transformed system by an energy conservation constraint in such a way that the resulting DAE system stays index-one throughout its trajectory.