CS 624: Numerical Solution of Differential Equations
Spring 2006
Problem Set 1

Handed out: Wed., Feb. 1.

Due: Fri., Feb. 10 in lecture.

1. Consider two LMS methods in standard form given by (1.2.11) in the text. Suppose their orders of accuracy are p_1, p_2. WLOG, assume they have the same value of s. Their sum, suitably rescaled to ensure that the leading α_j is 1, is also a LMS method. Let p be the order of the resulting LMS method.

   (a) Show that $p \geq \min(p_1, p_2)$.

   (b) Show that if $p_1 \neq p_2$, then $p = \min(p_1, p_2)$.

   (c) Give an example to show that if $p_1 = p_2$, then it may be possible that $p > \min(p_1, p_2)$. [Hint: there are two well known methods such that $p_1 = p_2 = 1$ but $p = 2$.]

2. Let
$$(x_1, y_1), \ldots, (x_n, y_n), (w_1, z_1)$$
   be a sequence of $n + 1$ real points in the plane such that $x_1 < x_2 < \cdots < x_n < w_1$. Assume $n > 0$. Show that there exists a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for $i = 1, \ldots, n$ and $p'(w_1) = z_1$. Note: this theorem is used to establish the validity of the interpolation-based definition of the BDF family.

   [Hint: As in lecture, first show uniqueness, then call upon linear algebra to conclude existence. To show uniqueness, first argue that p' is uniquely determined. In the uniqueness proof, Rolle's theorem will help you find $n - 1$ roots of the derivative of $p - q$, and there is already another root given.]

3. One difficulty with finite difference methods for IVP's is that they return approximations for u only at discrete time steps. A common technique for obtaining an approximation to u at other time-values is to interpolate between discrete points.

   Suppose $t_n = nk$, where k is a fixed stepsize. Suppose $v(t)$ is defined via linear interpolation, i.e., for $t \in [t_n, t_{n+1}]$, define
$$v(t) = \frac{t - t_n}{k} v^{n+1} + \frac{t_{n+1} - t}{k} v^n$$
   and use the approximation $u(t) \approx v(t)$.

   Determine the amount of additional error introduced by this interpolation (additional beyond the global truncation error of the LMS method) as a function of k. Explain why the error introduced by this interpolation formula is acceptable for AB1 and AB2 but probably not for AB-s when $s \geq 3$.

4. Consider a light body orbiting a heavy body located at the origin lying in a plane. The equation of motion is

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{\mathbf{x}}{\|\mathbf{x}\|^3}$$

where $\mathbf{x}(t) \in \mathbf{R}^2$ is the position of the light body. (The norm in the denominator is the 2-norm.) Convert this to a first-order system. (You should end up with a total of four dependent variables.) Write AB1 and AB2 algorithms in Matlab and apply them to this problem. Set up initial conditions in which the light body starts at $(1, 0)$ and is moving with velocity $(0, 1)$. Hand in plots of the trajectories of the bodies for the same initial condition for both AB1 and AB2, using two or three different time-step choices. Note: initialize AB2 with a single step of AB1.

Try an assortment of time-steps for AB1 and AB2. For each algorithm choice and each time step choice, determine the x-coordinate of the light body when it makes one full revolution (i.e., when the y-coordinate is 0 and the x-coordinate is positive). Note that there will probably not be an exact step when it lands on the positive x-axis again after $t = 0$, so you will need to determine this position using piecewise linear interpolation. If the system were behaving according to Newton's laws, the x-coordinate would be exactly 1 after one cycle (i.e., the body would return to its original point). Determine experimentally how the x-coordinate after one revolution depends on the time-step used for AB1 and AB2.

Turn in listings of your m-files, a paragraph of conclusions and at least one interesting plot.