CS 624: Numerical Solution of Differential Equations
Spring 2006
Prelim 2

Handed out: Tues., Mar. 28.

This exam has four questions. The questions are weighted equally. It counts for 20% of your final course grade (same as Prelim 1). This exam is due back at the end of lecture Friday, March 31 if you picked up the exam on Tuesday or at the end of lecture on Monday, April 3 if you picked up the exam on Friday.

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Trefethen then you must cite them.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until Tuesday, April 4. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

You are not allowed to send any email or otherwise make any on-line posting concerning the questions on this exam until after it is over. But you are allowed to consult publicly-available websites and search engines.

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in after lecture but before 5:00 p.m. on the due date will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are handed in on time.

1. (a) Let A be a 2×2 real matrix whose determinant is 1. Recall that the trace of A is $A(1,1) + A(2,2)$. Show that if the trace lies in $(-2,2)$, then eigenvalues of A are distinct and have magnitude equal to 1. Show that if the trace is greater than 2 or less than -2, then the eigenvalues of A are real, and one has magnitude greater than 1.

   (b) Use the result of (a) to show the following. Suppose the leapfrog method is applied to the simple harmonic oscillator $p' = -q$, $q' = p$. Then provided that $k < 2$, where k is the stepsize, the computed solution (p^n, q^n) will stay bounded as $n \to \infty$. If $k > 2$, then (p^n, q^n) will tend to infinity.

   (c) An interesting result about leapfrog in the literature (to be provided on the solution set) requires the following factorization. The iteration matrix A associated with leapfrog applied to the harmonic oscillator, i.e., the matrix A such that

$$\begin{pmatrix} p^{n+1} \\ q^{n+1} \end{pmatrix} = A \begin{pmatrix} p^n \\ q^n \end{pmatrix},$$

   can be factored as LQL^{-1}, where L is a lower triangular matrix whose $(1,1)$ entry is 1 and Q is an orthogonal matrix of the form $[c, s; -s, c]$ provided $k < 2$. Note that orthogonality implies $c^2 + s^2 = 1$. Give the details of this factorization (i.e., specify the entries of L and Q). Hint: The above factorization shows that Q is similar to A, and therefore Q and A have the same trace (why?) and same determinant (which is 1).

2. Consider the following linear semi-explicit DAE system:

$$\begin{aligned} \mathbf{y}' &= C\mathbf{x} + D\mathbf{y}, \\ A\mathbf{x} &= \mathbf{b}, \\ \mathbf{x}(0) = \mathbf{x}_0, &\qquad \mathbf{y}_0 = \mathbf{y}_0. \end{aligned}$$

   Here, $\mathbf{x}(t) \in \mathbf{R}^n$, $\mathbf{y}(t) \in \mathbf{R}^m$, and A is a square nonsingular $n \times n$ matrix, and $\mathbf{x}_0$ is not necessarily the solution to $A\mathbf{x} = \mathbf{b}$.

   Three possible ways to solve this are: (a) apply the backward Euler (BE) method directly to the DAE as described in lecture. (b) Solve the second equation for $\mathbf{x}$; substitute in the first equation, and apply BE to the resulting m-variable ODE system, or (c) consider a ODE system in which the constraint is treated as a penalty, that is, replace the constraint with the ODE $\mathbf{x}' = -\mu A^T(A\mathbf{x} - \mathbf{b})$ where $\mu > 0$ is very large.

   Argue that the first step of BE with stepsize k for methods (a) and (b) are equivalent. Argue that the first step of BE for (c) will yield a solution that is arbitrarily close to the result of BE for (a) or (b) as $\mu \to \infty$.

3. Consider a semidiscretization of the one-way wave equation $u_t = u_x$ given by $du_i/dt = (u_{i+1} - u_i)/h$, i.e., a forward difference approximation to the spatial derivative. Assume a Dirichlet condition is applied at the right end. Suppose this ODE is now temporally discretized with Euler's method.

   (a) Determine the upper limit on k (which may depend on h) for Euler's method to be stable for this ODE system. Note: it may be helpful to recall that the eigenvalues of an upper or lower triangular matrix are exactly the diagonal entries.

(b) The answer you obtained in part (a) is not the same as the CFL or Von-Neumann limit for upwind method for the original PDE. Argue that this discrepancy is not a mathematical contradiction by carefully stating the two definitions of stability and explaining how a time step can be stable for an ODE solver and yet not for the underlying PDE solver.

[Note: there are some papers in the literature that consider this discrepancy. I will give you a citation in the solutions to this prelim. One explanation for the discrepancy (but not relevant for answering the question) is that the vector $[1; -1; 1; -1; 1; -1 \ldots]$, though not an eigenvector of the Jacobian of the ODE, is "almost" an eigenvector, and its "pseudo-eigenvalue" is quite different from the true eigenvalues of the Jacobian.]

(c) Suppose the one-way wave equation is spatially discretized with backward differences. Show that the resulting ODE system is unstable, in the sense that solutions can grow exponentially with t.

4. Consider the Euler method

$$v_j^{n+1} = v_j^n + \frac{k}{h^2}(v_{j-1}^n - 2v_j^n + v_{j+1}^n)$$

for the diffusion equation $u_t = u_{xx}$. Assume a stable choice of step size. The Euler method is known to be first-order accurate. Show that the Euler method is actually second-order accurate for a different PDE $u_t = u_{xx} + \alpha u_{xxxx}$ where α is a scalar that you determine and that will depend on k and h.