CS 624: Numerical Solution of Differential Equations
Spring 2006
Prelim 1

Handed out: Tues., Feb. 21.
This exam has five questions. You have 90 minutes to answer all the questions. Write your answers in the booklet. You may consult a $8.5'' \times 11''$ piece of paper written on both sides that you have prepared in advance. The questions are weighted equally.

1. Consider integrating $u' = f(u,t)$, $u(0) = u_0$ using EM. Recall that the local truncation error, including all terms (not just the leading term), is $k^2u''(t)/2 + k^3u'''(t)/6 + \cdots + k^n u^{(n)}(t)/n! + \cdots$. (a) How is the global error $\|v^{1/k} - u(1)\|$ expected to depend on k assuming exact initialization? (b) Same question, except assuming that $\|v^0 - u_0\| = \text{const} \cdot k$? (c) Same two questions, except for the special case that $f(u,t) \equiv 1$?

2. Consider integrating the IVP $u' = f(u,t)$, $u(0) = u_0$ using an order-p LMS method. Suppose that the exact f is not available, but a program that computes f to within a tolerance ϵ is available (i.e., there is a program that takes $\epsilon > 0, u, t$ as input and returns $\tilde{f}$ such that $\|\tilde{f} - f(u,t)\| \le \epsilon$). How should ϵ relate to k (the time step) and p (the order of accuracy)? Explain.

3. Suppose the sequence v^n is computed as an approximate solution to $u' = f(u,t)$, $u(0) = u_0$, using AB-s with a fixed time step and sufficiently accurate initial conditions. Propose a method to approximate values of u at any time value t (not just $t_n = nk$) that is as accurate as AB-s. Note: you do not have to analyze the accuracy of your approximation; just write down the method. [Hint: Recall that the AB-s family of methods can be defined using an integral of an interpolant. This same integral can be used to define an approximation to u at other points besides nk.]

4. Sometimes it is useful to split an ODE into several terms and use a different method for each term, e.g., for $u'(t) = f(u) + g(u)$, we could carry out a EM/BE split as in: $v^{n+1} = v^n + kf(v^n) + kg(v^{n+1})$. For example, consider the IVP $u' = au + b(\sin u)^2$, $u(0) = u_0$ where $a, b < 0$ and $u_0 > 0$, a problem that has a solution monotonically decaying to 0. Suppose BE is applied to the linear term and EM to the nonlinear term. (a) Determine a stable choice of time step. Note that the choice may depend on the current value of v^n. (b) Explain why it might be useful to split the problem this way. [Hint: Contrast this method to other simpler methods based on the following two criteria: stability of the method and the ease of computing v^n.]

5. Consider a method that alternates two different LMS formulas:
$$v^{n+1} = \begin{cases} \alpha_1 v^n + \alpha_0 v^{n-1} + \beta_0 f^{n-1}k, & n \text{ even;} \\ \bar{\alpha}_1 v^n + \bar{\alpha}_0 v^{n-1} + \bar{\beta}_0 f^{n-1}k, & n \text{ odd.} \end{cases}$$
Write down a condition based on $\alpha_0, \bar{\alpha}_0, \alpha_1, \bar{\alpha}_1$ to ensure that the method is D-stable. A proof of D-stability is not necessary. [Hint: Find a 2×2 matrix A such that $[v^{n+2}; v^{n+1}] = A[v^n; v^{n-1}] + F$ for all even n's, where F is a term that depends on the β_j's and f^n's. It is suggested that the condition pertains to the eigenvalues of A.]