

1 Denotational Semantics of while-do

Last time, guided by the intuition that the programs `while b do c` and `if b then c ; while b do c else skip` should be equivalent, we defined the denotation of the statement `while b do c` as the least solution to the equation

$$W \triangleq \lambda\sigma \in \Sigma. \begin{cases} (W)^*(\mathcal{C}[[c]]\sigma), & \text{if } \mathcal{B}[[b]]\sigma, \\ \sigma, & \text{otherwise} \end{cases}$$

in $\Sigma \rightarrow \Sigma_{\perp}$; that is, the least fixpoint of the operator

$$\mathcal{F} \triangleq \lambda w \in \Sigma \rightarrow \Sigma_{\perp}. \lambda\sigma \in \Sigma. \begin{cases} (w)^*(\mathcal{C}[[c]]\sigma), & \text{if } \mathcal{B}[[b]]\sigma, \\ \sigma, & \text{otherwise} \end{cases}$$

of type $(\Sigma \rightarrow \Sigma_{\perp}) \rightarrow (\Sigma \rightarrow \Sigma_{\perp})$. More simply, we might write

$$\mathcal{F} \triangleq \lambda w \in \Sigma \rightarrow \Sigma_{\perp}. \lambda\sigma \in \Sigma. \text{if } \mathcal{B}[[b]]\sigma \text{ then } (w)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma$$

with the understanding that the if-then-else here is purely mathematical. Here if $w : \Sigma \rightarrow \Sigma_{\perp}$, then $(w)^* : \Sigma_{\perp} \rightarrow \Sigma_{\perp}$ is the *lift* of w , which sends \perp to \perp and x to $w(x)$ for $x \in \Sigma - \{\perp\}$.

In order to show that the least fixpoint of \mathcal{F} exists, we will apply the Knaster–Tarski theorem. However, we only proved the Knaster–Tarski theorem for the partial order of subsets of some universal set ordered by set inclusion \subseteq . We need to extend it to the more general case of chain-complete partial orders (CPOs). To apply this theorem, we must know that the function space $\Sigma \rightarrow \Sigma_{\perp}$ is a CPO and that \mathcal{F} is a continuous map on this space.

2 Chain-Complete Partial Orders and Continuous Functions

Recall that a binary relation \sqsubseteq on a set X is a *partial order* if it is

- *reflexive*: $x \sqsubseteq x$ for all $x \in X$;
- *transitive*: for all $x, y, z \in X$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$;
- *antisymmetric*: for all $x, y \in X$, if $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$.

It is a *total order* if for all $x, y \in X$, either $x \sqsubseteq y$ or $y \sqsubseteq x$.

If $A \subseteq X$, we say that x is an *upper bound* for A if $y \sqsubseteq x$ for all $y \in A$. We say that x is a *least upper bound* or *supremum* of A if

- x is an upper bound for A , and
- for all other upper bounds y of A , $x \sqsubseteq y$.

Upper bounds and suprema need not exist. For example, the set of natural numbers \mathbb{N} under its natural order \leq has no supremum in \mathbb{N} . However, if the supremum of any set exists, it is unique. A partially ordered set is said to be *complete* if all subsets have suprema. The supremum of a set C , if it exists, is denoted $\bigsqcup C$.

Note that all elements of X are (vacuously) upper bounds of the empty set \emptyset , so if the supremum of \emptyset exists, then it is necessarily the least element of the entire set. In this case we give it the name \perp .

A *chain* is a subset of X that is totally ordered by \sqsubseteq . For example, in the partial order of subsets of $\{0, 1, 2\}$ ordered by set inclusion, the set $\{\emptyset, \{2\}, \{1, 2\}, \{0, 1, 2\}\}$ is a chain. A partially ordered set is *chain-complete* if all nonempty chains have suprema. A chain-complete partially ordered set is called a CPO. The empty chain \emptyset is not included in the definition of chain-complete, but if the empty chain also has a supremum, then it is necessarily the least element \perp of the CPO. A CPO with a least element \perp is called *pointed*.

Let X and Y be CPOs (we use \sqsubseteq to denote the partial order in both X and Y). A function $f : X \rightarrow Y$ is *monotone* if f preserves order; that is, for all $x, y \in X$, if $x \sqsubseteq y$ then $f(x) \sqsubseteq f(y)$. For example, the exponential function $\lambda x. e^x : \mathbb{R} \rightarrow \mathbb{R}$ is monotone. A function $f : X \rightarrow Y$ is *continuous* if f preserves suprema of nonempty chains; that is, if $C \subseteq X$ is a nonempty chain in X , then $\bigsqcup_{x \in C} f(x)$ exists and equals $f(\bigsqcup C)$. Here $\bigsqcup_{x \in C} f(x)$ is alternate notation for $\bigsqcup \{f(x) \mid x \in C\}$.

Every continuous map is monotone: if $x \sqsubseteq y$, then $y = \bigsqcup \{x, y\}$, so by continuity $f(y) = f(\bigsqcup \{x, y\}) = \bigsqcup \{f(x), f(y)\}$, which implies that $f(x) \sqsubseteq f(y)$.

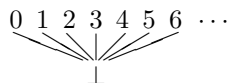
In the definition of continuity, we excluded the empty chain \emptyset . If it were included, then a continuous function would have to preserve \perp ; that is, $f(\perp) = \perp$. A continuous function that satisfies this property is called *strict*. We do not include \emptyset in the definition of continuous functions, because we wish to consider non-strict functions, such as the \mathcal{F} of Section 1.

3 The Knaster–Tarski Theorem in CPOs

Let $F : D \rightarrow D$ be any continuous function on a pointed CPO D . Then F has a least fixpoint $\text{fix } F \triangleq \bigsqcup_n F^n(\perp)$. The proof is a direct generalization of the proof for set operators given in an earlier lecture, where \perp was \emptyset and \bigsqcup was \bigcup . In a nutshell: by monotonicity, the $F^n(\perp)$ form a chain; since D is a CPO, the supremum $\text{fix } F$ of this chain exists; and by continuity, $\text{fix } F$ is preserved by F .

4 Flat Domains

Let S be a set with the *discrete ordering*, which means that any two distinct elements of S are \sqsubseteq -incomparable. We can make S into a pointed CPO S_\perp by adding a new bottom element \perp and defining $\perp \sqsubseteq \perp \sqsubseteq x \sqsubseteq x$ for all $x \in S$, but nothing else. This is called a *flat domain*. For example, \mathbb{N}_\perp looks like



Any flat domain is chain-complete, since every chain is finite, and every finite nonempty chain has a maximum element, which is its supremum.

5 Continuous Functions on CPOs Form a CPO

Now we claim that if C and D are CPOs, then the space of continuous functions $f : C \rightarrow D$ is a CPO under the pointwise ordering

$$f \sqsubseteq g \iff \forall x \in C \ f(x) \sqsubseteq g(x).$$

This space is denoted $[C \rightarrow D]$. It is easily verified that \sqsubseteq is a partial order on $[C \rightarrow D]$. If D is pointed with bottom element \perp , then $[C \rightarrow D]$ is also pointed with bottom element $\perp \triangleq \lambda x \in C. \perp$.

We need to show that $[C \rightarrow D]$ is chain-complete. Let \mathcal{C} be a nonempty chain in $[C \rightarrow D]$. Define

$$G \triangleq \lambda x \in C. \bigsqcup_{g \in \mathcal{C}} g(x).$$

First, G is a well-defined function, since for any $x \in C$, $\{g(x) \mid g \in \mathcal{C}\}$ is a chain in D , therefore its supremum $\bigsqcup_{g \in \mathcal{C}} g(x)$ exists. Also, the function G is continuous, since for any nonempty chain E in C ,

$$\begin{aligned} G(\bigsqcup E) &= \bigsqcup_{g \in \mathcal{C}} g(\bigsqcup E) && \text{by the definition of } G \\ &= \bigsqcup_{g \in \mathcal{C}} \bigsqcup_{x \in E} g(x) && \text{since each } g \in \mathcal{C} \text{ is continuous} \\ &= \bigsqcup_{x \in E} \bigsqcup_{g \in \mathcal{C}} g(x) && \text{by the lemma below} \\ &= \bigsqcup_{x \in E} G(x) && \text{again by the definition of } G. \end{aligned}$$

The third step in the above argument uses the following lemma.

Lemma 1. *If a_{xy} is a doubly-indexed collection of members of a partially ordered set such that*

- (i) *for all x , $\bigsqcup_y a_{xy}$ exists,*
- (ii) *for all y , $\bigsqcup_x a_{xy}$ exists, and*
- (iii) *$\bigsqcup_y \bigsqcup_x a_{xy}$ exists,*

then $\bigsqcup_x \bigsqcup_y a_{xy}$ exists and is equal to $\bigsqcup_y \bigsqcup_x a_{xy}$.

Proof. Clearly $\bigsqcup_y \bigsqcup_x a_{xy}$ is an upper bound for all a_{xy} , therefore it is an upper bound for all $\bigsqcup_y a_{xy}$; and if b is any other upper bound for all $\bigsqcup_y a_{xy}$, then $a_{xy} \sqsubseteq b$ for all x, y , therefore $\bigsqcup_y \bigsqcup_x a_{xy} \sqsubseteq b$, so $\bigsqcup_y \bigsqcup_x a_{xy}$ is the least upper bound for all $\bigsqcup_y a_{xy}$; that is, $\bigsqcup_x \bigsqcup_y a_{xy} = \bigsqcup_y \bigsqcup_x a_{xy}$. \square

To apply this lemma, we need to know that

- (i) for all $g \in \mathcal{C}$, $\bigsqcup_{x \in E} g(x)$ exists,
- (ii) for all $x \in E$, $\bigsqcup_{g \in \mathcal{C}} g(x)$ exists, and
- (iii) $\bigsqcup_{g \in \mathcal{C}} \bigsqcup_{x \in E} g(x)$ exists.

But (i) holds because all $g \in \mathcal{C}$ are continuous, therefore $\bigsqcup_{x \in E} g(x) = g(\bigsqcup E)$; (ii) holds because $\{g(x) \mid g \in \mathcal{C}\}$ is a chain in D , and D is chain-complete; and (iii) follows from (i) and (ii) by taking $x = \bigsqcup E$.

6 Fixpoints and the Semantics of while-do

Now let us return to the denotational semantics of the while loop. We previously defined the function

$$\begin{aligned} \mathcal{F} & : (\Sigma \rightarrow \Sigma_{\perp}) \rightarrow (\Sigma \rightarrow \Sigma_{\perp}) \\ \mathcal{F} & \triangleq \lambda w \in \Sigma \rightarrow \Sigma_{\perp}. \lambda \sigma \in \Sigma. \text{if } \mathcal{B}[[b]]\sigma \text{ then } (w)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma. \end{aligned}$$

Any function $\Sigma \rightarrow \Sigma_{\perp}$ is continuous, since chains in the discrete space Σ contain at most one element, thus the space of functions $\Sigma \rightarrow \Sigma_{\perp}$ is the same as the space of continuous functions $[\Sigma \rightarrow \Sigma_{\perp}]$. Moreover, the lift $(w)^* : \Sigma_{\perp} \rightarrow \Sigma_{\perp}$ of any function $w : \Sigma \rightarrow \Sigma_{\perp}$ is continuous.

By previous arguments, the function space $[\Sigma \rightarrow \Sigma_{\perp}]$ is a pointed CPO, and \mathcal{F} maps this space to itself. To obtain a least fixpoint by Knaster–Tarski, we need to know that \mathcal{F} is continuous.

Let us first check that it is monotone. This will ensure that, when trying to check the definition of continuity, when C is a chain, $\{\mathcal{F}(d) \mid d \in C\}$ is also a chain, so that $\bigsqcup_{d \in C} \mathcal{F}(d)$ exists. Suppose $d \sqsubseteq d'$. We want to show that $\mathcal{F}(d) \sqsubseteq \mathcal{F}(d')$. But for all σ ,

$$\begin{aligned} \mathcal{F}(d)(\sigma) & = \text{if } \mathcal{B}[[b]]\sigma \text{ then } (d)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma \\ & \sqsubseteq \text{if } \mathcal{B}[[b]]\sigma \text{ then } (d')^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma \\ & = \mathcal{F}(d')(\sigma). \end{aligned}$$

Here we have used the fact that the operator $(\cdot)^*$ is monotone, which is easy to check.

Now let us check that \mathcal{F} is continuous. Let C be an arbitrary chain. We want to show that $\bigsqcup_{d \in C} \mathcal{F}(d) = \mathcal{F}(\bigsqcup C)$. We have

$$\begin{aligned} \bigsqcup_{d \in C} \mathcal{F}(d) & = \bigsqcup_{d \in C} \lambda \sigma. \text{if } \mathcal{B}[[b]]\sigma \text{ then } (d)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma \\ & = \lambda \sigma. \bigsqcup_{d \in C} \text{if } \mathcal{B}[[b]]\sigma \text{ then } (d)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma \\ & = \lambda \sigma. \text{if } \mathcal{B}[[b]]\sigma \text{ then } \bigsqcup_{d \in C} (d)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma \\ & = \lambda \sigma. \text{if } \mathcal{B}[[b]]\sigma \text{ then } (\bigsqcup C)^*(\mathcal{C}[[c]]\sigma) \text{ else } \sigma = \mathcal{F}(\bigsqcup C), \end{aligned}$$

since $\mathcal{B}[[b]]\sigma$ does not depend on d and since the lift operator $(\cdot)^*$ is continuous.