1   Review

Goal: We want to define $\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!]$ to be the least fixed point of Γ, where

$$\Gamma = \lambda f \in \Sigma_\perp \to \Sigma_\perp.\, \lambda\sigma \in \Sigma_\perp.\, if\ \neg\mathcal{B}[\![b]\!]\ then\ \sigma\ else\ f(\mathcal{C}[\![c]\!]\sigma).$$

Suggestion: Since the functions $\Gamma^n(\perp)$ approximate the desired result, it might be a good idea to let

$$\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] = \bigsqcup_{n\in\omega} \Gamma^n(\perp).$$

Problems:

- The least upper bound might not exist. $\Sigma_\perp \to \Sigma_\perp$ is a cpo, which ensures that every chain has a LUB. What if the functions $\Gamma^n(\perp)$ do not form a chain?

- Assuming the LUB exists, it might not be a fixed point of Γ, let alone the least one.

Solution: We introduce some conditions under which the LUB exists and equals the least fixed point, and then we show that these conditions hold for our Γ.

2   Monotonicity and Continuity

Definition: Let $(D, \sqsubseteq)$ be a cpo, $\Phi : D \to D$ a function. Φ is *monotonic* if

$$\forall x, y \in D \quad x \sqsubseteq y \to \Phi(x) \sqsubseteq \Phi(y).$$

Claim: If $(D, \sqsubseteq, \perp)$ is a pointed cpo and $\Phi : D \to D$ is monotonic then the elements $\Phi^n(\perp)$ form an increasing chain in D:

$$\perp \sqsubseteq \Phi(\perp) \sqsubseteq \Phi^2(\perp) \sqsubseteq \ldots$$

Proof: Since $\perp$ is the least element of D, we have

$$\perp \sqsubseteq \Phi(\perp).$$

Monotonicity of Φ gives

$$\forall n \in \omega \quad \Phi^n(\perp) \sqsubseteq \Phi^{n+1}(\perp) \to \Phi^{n+1}(\perp) \sqsubseteq \Phi^{n+2}(\perp).$$

The result follows by induction.

Notice that if $\Phi : D \to D$ is monotonic and $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ is a chain in D, then $\Phi(x_0) \sqsubseteq \Phi(x_1) \sqsubseteq \Phi(x_2) \sqsubseteq \ldots$ is also a chain in D. This permits the following definition.

Definition: Let $(D, \sqsubseteq)$ be a cpo, $F : D \to D$ a monotonic function. Φ is *continuous* if for every chain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$$

in D, Φ preserves the LUB operator:

$$\bigsqcup_{n\in\omega} \Phi(x_n) \quad = \quad \Phi(\bigsqcup_{n\in\omega} x_n).$$

3 The Fixed Point Theorem

We will now show that the properties of monotonicity and continuity allow us to compute the least fixed point as desired.

Claim: Let $(D, \sqsubseteq)$ be a cpo, and let $\Phi : D \to D$ be a monotonic continuous function. Then $\bigsqcup_{n \in \omega} \Phi^n(\bot)$ is a fixed point of Φ.

Proof: By continuity of Φ,

$$\Phi(\bigsqcup_{n \in \omega} \Phi^n(\bot)) = \bigsqcup_{n \in \omega} \Phi(\Phi^n(\bot))$$

Applying Φ,

$$= \bigsqcup_{n \in \omega} \Phi^{n+1}(\bot)$$

Reindexing,

$$= \bigsqcup_{n=1,2,\dots} \Phi^n(\bot)$$

By definition of $\bot$,

$$= \bot \sqcup \bigsqcup_{n=1,2,\dots} \Phi^n(\bot)$$

And, finally, absorbing the join with $\bot$ into the big join,

$$= \bigsqcup_{n \in \omega} \Phi^n(\bot)$$

We now know that monotonicity and continuity guarantee that $\bigsqcup_{n \in \omega} \Phi^n(\bot)$ is a fixed point of Φ. We also want $\bigsqcup_{n \in \omega} \Phi^n(\bot)$ to be the *least* fixed point of Φ. To show this, we must prove that $y = \Phi(y) \Rightarrow \bigsqcup_{n \in \omega} \Phi^n(\bot) \sqsubseteq y$. We can actually prove something even stronger.

Definition: Let $(D, \sqsubseteq)$ be a cpo, $\Phi : D \to D$ a function. $x \in D$ is a *prefixed* point of Φ if $\Phi(x) \sqsubseteq x$.

Notice that every fixed point of Φ is also a prefixed point. As a consequence, if a fixed point of Φ is the least prefixed point of Φ, it is also the least fixed point of Φ.

Claim: Let $(D, \sqsubseteq, \bot)$ be a pointed cpo. For any monotonic continuous function, $\Phi : D \to D$, $\bigsqcup_{n \in \omega} \Phi^n$ is the least prefixed point of Φ.

Proof: Suppose y is a prefixed point of Φ. By definition of $\bot$,

$$\bot \sqsubseteq y$$

Taking Φ of both sides,

$$\Phi(\bot) \sqsubseteq \Phi(y)$$

Inductively, for all $n \geq 0$,

$$\Phi^n(\bot) \sqsubseteq \Phi^n(y)$$

Taking the join of each of these sequences yields:

$$\bigsqcup_{n \in \omega} \Phi^n(\bot) \sqsubseteq \bigsqcup_{n \in \omega} \Phi^n(y)$$

and thus,

$$\bigsqcup_{n \in \omega} \Phi^n(\bot) \sqsubseteq y$$

We have now proven:

The Fixed Point Theorem: Let $(D, \sqsubseteq, \bot)$ be a pointed cpo. For any monotonic continuous function, $\Phi : D \to D$, $\bigsqcup_{n \in \omega} \Phi^n$ is the least fixed point of Φ.

We have actually encountered the fixed point theorem before. Recall lecture 6, when we defined the set of all elements derivable in some rule system to be the least fixed point of the rule operator, R. Our proof in that case was an instantiation of the fixed point theorem on the CPO consisting of all subsets of a set, ordered by set inclusion:

$$R = \Gamma$$

$$\emptyset = \bot$$

$$\bigcup = \bigsqcup$$

$$\subseteq = \sqsubseteq$$

Note that the tricky part of the earlier proof corresponded to showing that R is a continuous operator, which was true because we only allow inference rules with a finite number of premises.

4   Final Thoughts:

One may be sensing that our quest to define a denotational semantics for IMP has led us to more serious mathematics than we encountered in defining the operational semantics. This may be attributed to the fact that we are attempting to map from syntax to meaning, while maintaining order and ensuring that two equivalent pieces of syntax are always mapped to the same meaning. This leads to some deeper mathematical structure, including the equivalence classes of syntax.

Note that we have not yet defined a full denotational semantics for IMP. We have not yet shown that our function, $\Gamma = \lambda f . \lambda \sigma . if\ (\sigma = \bot)\ then\ \bot\ else\ if\ \neg \mathcal{B}[\![b]\!]\sigma\ then\ \sigma\ else\ f(\mathcal{C}[\![b]\!]\sigma)$ is monotonic and continuous, so we cannot yet claim the correctness of our definition of $\mathcal{C}[\![\mathsf{while\ b\ do\ c}]\!]$. We could easily verify that our Γ does indeed satisfy these properties. However, if we later decide to change any part of our definitions of any commands (c) or boolean expressions (b), we would have to prove that the resulting modified Γ still satisfied these properties. So, a more general method is desirable. We will define a set of constructs that always yield monotonic continuous functions. Stay tuned...