Problem Set 10

Due Date: Thurs, April 24, 2003

Problems

1. Show how to express the following functions as μ-recursive functions.
   (a) The unary constant $const_k$ with $const_k(x) = k$ for arbitrary $k \in \mathbb{N}$
   (b) Exponentiation $exp(x, y) = x^y$
   (c) Sum of a list of function values $Sum_f(y) = \sum_{i=0}^{y} f(i)$

   To express a function you may only use the functions s, c_k, and π_i^n; the operations $\circ$ (composition), pr (primitive recursion), and μ (minimization), and symbols for auxiliary functions that you prove to be μ-recursive.

2. Show how to represent the following functions in Peano Arithmetic
   (a) Division div with $div(x, y) = x \div y$
   (b) The function $divides$ with $divides(x, y) = \begin{cases} 1 & \text{if } x \text{ divides } y \\ 0 & \text{otherwise} \end{cases}$
   (c) The function $prime$ with $prime(x) = \begin{cases} 1 & \text{if } x \text{ is a prime number} \\ 0 & \text{otherwise} \end{cases}$

3. Prove $(\forall x)(x{+}1 \neq x)$ in Peano Arithmetic

4. Show by providing an appropriate model that the following laws are not valid in $\mathcal{Q}$
   (a) $(\forall x, y)\ (x + y = y + x)$
   (b) $(\forall x, y, z)\ (x + (y + z) = (x + y) + z)$
   (c) $(\forall x)\ (0 + x = x)$
   (d) $(\forall x, y)\ (x * y = y * x)$
   (e) $(\forall x)\ (0 * x = 0)$

5. **Extra credit.** The function $A : \mathbb{N}^2 \to \mathbb{N}$ is defined recursively as follows
   - $A(0, 0) = 1,\ \ A(0, 1) = 2,\ \ A(0, y) = y{+}2$ otherwise
   - $A(n{+}1, 0) = 1,\ \ A(n{+}1, y{+}1) = A(n, A(n{+}1, y))$

   Show that A is μ-recursive and calculate $A(4, 4)$.