CS486 Problem Set 10 DUE: 4/24/03

1. We show how to express the following functions as μ-recursive functions.

   (a)

   $$const_k(x) = k \qquad \text{for arbitrary } k \in \mathbb{N}$$

   Define

   $$const_k \equiv \mathsf{pr}(c_k, \pi_2^2)$$

   (b)

   $$exp(x, y) = x^y$$

   Define

   $$add \equiv \mathsf{pr}(\pi_1^1, s \circ \pi_3^3)$$

   Define

   $$mul \equiv \mathsf{pr}(const_0, add \circ \pi_1^3, \pi_3^3)$$

   Define

   $$exp \equiv \mathsf{pr}(const_1, mul \circ \pi_1^3, \pi_3^3)$$

   (c)

   $$Sum_f(y) = \sum_{i=0}^{y} f(i)$$

   Define

   $$Sum_f \equiv \mathsf{pr}(f \circ c_o, add \circ (f \circ s \circ \pi_1^2), \pi_2^2)$$

2. We show how to express the following functions in Peano arithmetic.

   First, define

$$x < y \equiv (\exists z)(x + (z + \mathbf{1}) = y)$$

$$x \leq y \equiv (x < y) \vee (x = y)$$

$$x | y \equiv (\exists z)(x * z = y)$$

   (a)

$$div(x, y) = x \div y$$

   Define

$$R_{div}(x, y, z) \equiv (\exists a)((a < y) \wedge (x = z * y + a))$$

   (b)

$$divides(x, y) = \begin{cases} 1 & \text{if } x \text{ divides } y \\ 0 & \text{otherwise} \end{cases}$$

   Define

$$R_{divides}(x, y, z) \equiv ((x|y) \supset (z = \mathbf{1})) \wedge (\sim(x|y) \supset (z = \mathbf{0}))$$

   (c)

$$prime(x) = \begin{cases} 1 & \text{if } x \text{ is a prime number} \\ 0 & \text{otherwise} \end{cases}$$

   Define

$$\begin{aligned} R_{prime}(x, z) \equiv \quad & (((\forall a)((a|x) \supset ((a = \mathbf{1}) \vee (a = x)))) \supset (z = \mathbf{1})) \\ & \wedge \\ & (((\exists a)((a|x) \wedge \sim(a = \mathbf{1}) \wedge \sim(a = x))) \supset (z = \mathbf{0})) \end{aligned}$$

3. We prove $(\forall x)(\sim(x+\mathbf{1}=x))$ in Peano arithmetic.

We first prove a useful lemma:

`lemma1`: $(\forall x,y)((x=y)\supset(x+\mathbf{1}=y+\mathbf{1}))$

	$x=y$	
$\wedge$	$x+(\mathbf{0}+\mathbf{1})=(x+\mathbf{0})+\mathbf{1}$	`add-step`[x,$\mathbf{0}$]
$\supset$	$x+(\mathbf{0}+\mathbf{1})=(y+\mathbf{0})+\mathbf{1}$	`subst`
$\supset$	$(x+\mathbf{0})+\mathbf{1}=(y+\mathbf{0})+\mathbf{1}$	`assoc`$_+$[x,$\mathbf{0}$,$\mathbf{1}$],`subst`
$\supset$	$x+\mathbf{1}=(y+\mathbf{0})+\mathbf{1}$	`add-base`[x],`subst`
$\supset$	$x+\mathbf{1}=y+\mathbf{1}$	`add-base`[y],`subst`

`thm`: $(\forall x)(\sim(x+\mathbf{1}=x))$

		`base`[x]
	$\sim(\mathbf{0}+\mathbf{1}=\mathbf{0})$	`non-surjective`[$\mathbf{0}$]
	$\sim(x+\mathbf{1}=x)$	`ihyp`[x]
		`step`[x]
	$\sim(x+\mathbf{1}=x)$	`ihyp`[x]
$\supset$	$\sim((x+\mathbf{1})+\mathbf{1}=x+\mathbf{1})$	`lemma1`[$x+\mathbf{1}$,x],`subst`

4. We show that the following laws are not valid in $\mathcal{Q}$.

   (a) $(\forall x, y)(x + y = y + x)$
   (b) $(\forall x, y, z)(x + (y + z) = (x + y) + z)$
   (c) $(\forall x)(\mathbf{0} + x = x)$
   (d) $(\forall x, y)(x * y = y * x)$
   (e) $(\forall x)(\mathbf{0} * x = \mathbf{0})$

   Consider the following model of $\mathcal{Q}$:

$$U = \mathbb{N} \cup \{\omega, \zeta\}$$

$+$	j	ω	ζ
i	$i + j$	ζ	ω
ω	ω	ζ	ω
ζ	ζ	ζ	ω

$*$	$\mathbf{0}$	j	ω	ζ
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	ω	ζ
i	$\mathbf{0}$	$i * j$	ω	ζ
ω	$\mathbf{0}$	ζ	ζ	ζ
ζ	$\mathbf{0}$	ω	ω	ω

   We show that the following laws are not valid in this model:

   (a) $(\forall x, y)(x + y = y + x)$

$$\omega + \mathbf{0} = \omega \neq \zeta = \mathbf{0} + \omega$$

   (b) $(\forall x, y, z)(x + (y + z) = (x + y) + z)$

$$\omega + (\mathbf{0} + \omega) = \omega + \zeta = \omega \neq \zeta = \omega + \omega = (\omega + \mathbf{0}) + \omega$$

   (c) $(\forall x)(\mathbf{0} + x = x)$

$$\mathbf{0} + \omega = \zeta \neq \omega$$

   (d) $(\forall x, y)(x * y = y * x)$

$$\omega * \mathbf{0} = \mathbf{0} \neq \omega = \mathbf{0} * \omega$$

   (e) $(\forall x)(\mathbf{0} * x = \mathbf{0})$

$$\mathbf{0} * \omega = \omega \neq \mathbf{0}$$

We show that the axioms of $\mathcal{Q}$ are satisfied by this model:

`non-surjective`:

$$\omega + \mathbf{1} = \zeta \neq \mathbf{0} \qquad \zeta + \mathbf{1} = \omega \neq \mathbf{0}$$

`injective`:

$$\begin{array}{ll} & \omega + \mathbf{1} = y + \mathbf{1} \\ \supset & \omega = y + \mathbf{1} \\ \supset & \omega = y \end{array} \qquad \begin{array}{ll} & x + \mathbf{1} = \omega + \mathbf{1} \\ \supset & x + \mathbf{1} = \omega \\ \supset & x = \omega \end{array}$$

$$\begin{array}{ll} & \zeta + \mathbf{1} = y + \mathbf{1} \\ \supset & \zeta = y + \mathbf{1} \\ \supset & \zeta = y \end{array} \qquad \begin{array}{ll} & x + \mathbf{1} = \zeta + \mathbf{1} \\ \supset & x + \mathbf{1} = \zeta \\ \supset & x = \zeta \end{array}$$

`nonzero`:

$$\begin{array}{ll} & \omega + \mathbf{1} = \omega \\ \supset & (\exists z)(z + \mathbf{1} = \omega) \end{array} \qquad \begin{array}{ll} & \zeta + \mathbf{1} = \zeta \\ \supset & (\exists z)(z + \mathbf{1} = \zeta) \end{array}$$

`add-base`:

$$\omega + \mathbf{0} = \omega \qquad \zeta + \mathbf{0} = \zeta$$

`add-step`:

$$\begin{gathered} \omega + (j + \mathbf{1}) = \omega = \omega + \mathbf{1} = (\omega + j) + \mathbf{1} \\ \omega + (\omega + \mathbf{1}) = \omega + \omega = \zeta = \zeta + \mathbf{1} = (\omega + \omega) + \mathbf{1} \\ \omega + (\zeta + \mathbf{1}) = \omega + \zeta = \omega = \omega + \mathbf{1} = (\omega + \zeta) + \mathbf{1} \\ \zeta + (j + \mathbf{1}) = \zeta = \zeta + \mathbf{1} = (\zeta + j) + \mathbf{1} \\ \zeta + (\omega + \mathbf{1}) = \zeta + \omega = \zeta = \zeta + \mathbf{1} = (\zeta + \omega) + \mathbf{1} \\ \zeta + (\zeta + \mathbf{1}) = \zeta + \zeta = \omega = \omega + \mathbf{1} = (\zeta + \zeta) + \mathbf{1} \end{gathered}$$

`mul-base`:

$$\omega * \mathbf{0} = \mathbf{0} \qquad \zeta * \mathbf{0} = \mathbf{0}$$

`mul-step`:

$$\begin{gathered} \omega * (j + \mathbf{1}) = \zeta = \zeta + \omega = \omega * j + \omega \\ \omega * (\omega + \mathbf{1}) = \omega * \omega = \zeta = \zeta + \omega = \omega * \omega + \omega \\ \omega * (\zeta + \mathbf{1}) = \omega * \zeta = \zeta = \zeta + \omega = \omega * \zeta + \omega \\ \zeta * (j + \mathbf{1}) = \omega = \omega + \zeta = \zeta * j + \zeta \\ \zeta * (\omega + \mathbf{1}) = \zeta * \omega = \omega = \omega + \zeta = \zeta * \omega + \zeta \\ \zeta * (\zeta + \mathbf{1}) = \zeta * \zeta = \omega = \omega + \zeta = \zeta * \zeta + \zeta \end{gathered}$$

5. The function $A : \mathbb{N}^2 \to \mathbb{N}$ is defined recursively as follows:

$$\begin{array}{rcll} A(0,0) & = & 1 & \\ A(0,1) & = & 2 & \\ A(0,y) & = & y+2 & \text{otherwise} \\ A(n+1,0) & = & 1 & \\ A(n+1,y+1) & = & A(n, A(n+1,y)) & \end{array}$$

We show that A is μ-recursive.

First, define the function $F : \mathbb{N}^3 \to \mathbb{N}$ recursively as follows:

$$\begin{array}{rcl} F(n,0,x) & = & 0 \\ F(n,m,0) & = & m \\ F(n,m,x+1) & = & A(n-x+1, F(n,m,x)-1) \end{array}$$

We note that the evaluation of $A(n,m)$ only requires the evaluation of $A(x,y)$ where $0 \leq x \leq n$ and $0 \leq y \leq F(n,m,x)$.

Recall that $\langle\rangle : \mathbb{N}^2 \to \mathbb{N}$ is the bijective encoding of pairs of numbers, defined as $\langle i,j\rangle \equiv j + (i+j)(i+j+1)/2$. Recall that $\star : \mathbb{N}^2 \to \mathbb{N}$ is the selection function, which yields y_i for $\hat{y} \star i$.

Define

$$p \equiv \mathsf{pr}(c_0, \pi^2_1)$$

$$sub \equiv \mathsf{pr}(\pi^1_1, p \circ \pi^2_2)$$

$$F \equiv pr(const_0 \circ \pi^3_3, \mathsf{pr}(\pi^3_2, \star \circ (sub \circ (s \circ \pi^5_1), \pi^5_4), (p \circ \pi^5_5)) \circ \pi^5_1, (s \circ \pi^5_4), \pi^5_2, \pi^5_3) \circ \pi^4_1, \pi^4_3, \pi^4_4, \pi^4_2$$

$$A(n,m) \equiv \mathsf{min}\left\{ z \;\middle|\; \begin{array}{c} (z \star \langle 0,0\rangle = 1) \wedge \\ (z \star \langle 0,1\rangle = 2) \wedge \\ (\forall y)\left(\begin{array}{c} ((1 < y) \wedge (y \leq F(n,m,z,0))) \supset \\ (z \star \langle 0,y\rangle = s(s(y))) \end{array} \right) \wedge \\ (\forall x)((x < n) \supset (z \star \langle s(x),0\rangle = 1)) \wedge \\ (\forall x,y)\left(\begin{array}{c} ((x < n) \wedge (y \leq F(n,m,z,s(x)))) \supset \\ (z \star \langle s(x),s(y)\rangle = z \star \langle x, z \star \langle s(x),y\rangle\rangle) \end{array} \right) \end{array} \right\} \star \langle n,m\rangle$$

We calculate $A(4,4)$.

We first prove the following:

$$A(0,0) = 1 \quad A(0,1) = 2 \quad A(0,y) = y+2$$

Obvious from the definition of A.

$$A(1,0) = 1 \quad A(1,y) = 2 * y$$

Proceed by induction on y. Note $A(1,0) = 1$ by the definition of A. Note $A(1,1) = A(0, A(1,0)) = A(0,1) = 2$ by the definition of A. Suppose $A(1,y) = 2 * y$ for $1 \leq y$. Note $A(1, y+1) = A(0, A(1,y)) = A(0, 2*y) = 2*y+2 = 2*(y+1)$ for $1 \leq y$.

$$A(2,y) = 2^y$$

Proceed by induction on y. Note $A(2,0) = 1$ by the definition of A. Suppose $A(2,y) = 2^y$ for $0 \leq y$. Note $A(2,y+1) = A(1, A(2,y)) = A(1, 2^y) = 2 * 2^y = 2^{y+1}$ for $0 \leq y$.

$$A(3,y) = 2 \uparrow y = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{y \text{ times}}$$

Proceed by induction on y. Note $A(3,0) = 1$ by the definition of A. Suppose $A(3,y) = 2 \uparrow y$ for $0 \leq y$. Note $A(3,y+1) = A(2, A(3,y)) = A(2, 2 \uparrow y) = 2^{2 \uparrow y} = 2 \uparrow (y+1)$ for $0 \leq y$.

$$A(4,y) = 2 \uparrow\uparrow y = \underbrace{2 \uparrow (2 \uparrow (\cdots \uparrow 2))}_{y \text{ times}}$$

Proceed by induction on y. Note $A(4,0) = 1$ by the definition of A. Suppose $A(4,y) = 2 \uparrow\uparrow y$ for $0 \leq y$. Note $A(4,y+1) = A(3, A(4,y)) = A(3, 2 \uparrow\uparrow y) = 2 \uparrow (2 \uparrow\uparrow y) = 2 \uparrow\uparrow (y+1)$ for $0 \leq y$.

Hence

$$\begin{aligned}
A(4,4) &= 2 \uparrow\uparrow 4 \\
&= 2 \uparrow (2 \uparrow (2 \uparrow 2)) \\
&= 2 \uparrow \left(2 \uparrow \left(2^2\right)\right) \\
&= 2 \uparrow (2 \uparrow 4) \\
&= 2 \uparrow \left(2^{2^{2^2}}\right) \\
&= 2 \uparrow \left(2^{2^4}\right) \\
&= 2 \uparrow \left(2^{16}\right) \\
&= 2 \uparrow 65536
\end{aligned}$$