Problem Set 9

Due Date: Thurs, April 17, 2003

Problems

1. a) Give a finite model of a semigroup that is not commutative.

   b) Give a finite model of a commutative semigroup that is not a monoid.

2. Consider the boolean ring $\langle \mathbb{B}, =, \Leftrightarrow, \vee, \mathsf{T}, \mathsf{F} \rangle$. Define the operations $\sim$, $\wedge$, and $\supset$ in terms of the ring operations and prove the followng laws solely on the basis of the ring axioms.

   (1) $p \supset (p \vee q)$,
   (2) $(p \wedge q) \supset p$,
   (3) $(p \wedge q) \supset q$,
   (4) $p \supset (q \supset p)$,
   (5) $\sim q \supset (q \supset p)$,
   (6) $p \supset q \supset (\sim q \supset \sim p)$, and
   (7) $p \vee p \Leftrightarrow p$.

3. Is $\langle \mathbb{Z}, =_5, +, * \rangle$ a field?
   If so, give brief proofs of the axioms. If not, show which axiom is not satisfied.

4. Define $x<y \equiv (\exists z)(x+z+1 = y)$ and prove the seven axioms of discrete linear orders for $<$ from the Peano axioms.

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lt-asym:     (∀x,y) (x<y ⊃ ∼(y<x))
lt-trans:    (∀x,y,z) ((x<y ∧ y<z) ⊃ x<z)
lt-linear:   (∀x,y) (x<y ∨ y<x ∨ x=y)
lt-discrete: (∀x,y) ∼(x<y ∧ y<x+1)
lt-0-1:      0<1
lt-mono-+:   (∀x,y,z)(x<y ⊃ x+z < y+z)
lt-mono-*:   (∀x,y,z)((0<z ∧ x<y) ⊃ x*z < y*z)
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