CS 486

Sample Solution for Problem #1, Assignment 3

1. Suppose S is a set such that for every interpretation v_0 there is always at least one element of S which is true under v_0. Show that there is a finite subset $\{X_1, \ldots, X_n\}$ of S such that the disjunction $(X_1 \vee \cdots \vee X_n)$ is a tautology.

   There were many logical mistakes and incorrect assumptions made on this problem. The three most common errors were:

   (a) Assuming S is finite.

   If S is finite then the problem is trivial, because then the finite set containing all of S satisfies the conclusion under the assumption that every valuation v_0 makes at least one element of S true. The hint suggested that you use compactness, which wouldn't be necessary if S were finite. There's no good reason S should be finite in this problem.

   (b) Assuming that the number of different truth valuations that apply to S is finite.

   This was also an unwarranted assumption. It might seem reasonable if you also assumed S is finite, but if S is infinite then the formulas in S could contain an infinite number of different variables, each with 2 different truth values, and so there would be an infinite number of different truth valuations.

   (c) Incorrectly applying the Compactness Theorem.

   The Compactness Theorem states that "if every finite subset of a set S of formulas is satisfiable, then the entire set S is satisfiable". If set S were finite, then this would be trivial to prove since every finite set is a finite subset of itself. However, the Compactness Theorem does not say anything directly about the situation in this problem. In this problem we are given that every valuation satisfies some member of S, which is not equivalent to the conditions for the Compactness Theorem.

   These were all logical errors. It's always a good idea to check any proof (either one you're writing or one of someone else's that you're reading) by asking yourself: Do all the facts I used follow immediately from the problem statement? Have I made any assumptions that are not justified? Have I used any theorems or lemmas correctly? This is a good way to make sure your proof is correct, or to make sure you understand someone else's proof. Even if you don't know how to fix the proof, at least then you can ask for help.

   This concludes the sermon. Now on to the solution.

   We will prove the statement above by contradiction, by assuming that every interpretation v_0 makes some element of S true, but that there is no finite subset $\{X_1, \ldots, X_n\}$ such that $(X_1 \vee \cdots \vee X_n)$ is a tautology. Under these assumptions, observe that every finite subset F of the set $T = \{\neg X \mid X \in S\}$ is of the form $\{\neg X_1, \ldots, \neg X_n\}$, where $\{X_1, \ldots, X_n\} \subseteq S$. So F is satisfiable if and only if the formula $(\neg X_1 \wedge \cdots \wedge \neg X_n)$ is satisfiable. Now by De Morgan's laws, this formula is equivalent to $\neg(X_1 \vee \cdots \vee X_n)$, which is the negation of $(X_1 \vee \cdots \vee X_n)$, where $\{X_1, \ldots, X_n\} \subseteq S$. Since we assumed that no such subsets were tautologies, it must be the case that $(X_1 \vee \cdots \vee X_n)$ is false under some interpretation v. Then $\neg(X_1 \vee \cdots \vee X_n)$ is true under v so $(\neg X_1 \wedge \cdots \wedge \neg X_n)$ is also true under v. Hence $(\neg X_1 \wedge \cdots \wedge \neg X_n)$ is satisfiable. This is true for any finite subset F of T, so every finite subset F of T is satisfiable and so by the Compactness Theorem (applied to T), T is satisfiable. Let v_0 be a satisfying interpretation for all of T. Then v_0 makes every formula of T true. But then every formula of S is false under v_0, since each formula of S is the negation of a formula of T. This contradicts the assumption that every truth assignment makes at least one formula of S true, so we have a contradiction.

This proof is intentionally very careful to make each logical step explicit. You don't have to be this careful, as long as you don't miss any major steps. It might help to try to do proofs as carefully as possible (if you have time) until you get the hang of it.