CS 486 Assignment 5
Sample Solution for Problem #1, part 3 (Smullyan p. 50 ex. 3)

For this problem it was not enough to give the formula $F = (\forall x)(\exists y)[x < y]$, because this allows $<$ to be interpreted as almost any relation, not just the "usual" $<$ which is a total order on the natural numbers. Remember, $<$ is just a symbol from a logical point of view. So the answer $G = (\forall x)(\exists y)[Pxy]$ is actually the same as F because we don't know any more about P from this formula than we do about $<$ from F. And we run into the same problem with the answer $(\forall x)(\exists y)[y = x + 1]$ because we don't know anything else about $+$ from this formula.

The problem asked for a formula which is satisfiable in some denumerably infinite universe but *not satisfiable* in any finite universe. This means that F must be false in any finite universe, under *any* interpretation, not just the particular interpretation where U is a finite subset of $\mathbb{N}$ and $<$ is the usual order restricted to U.

For example, for any universe, finite or denumerably infinite, P (or $<$) could be interpreted as the equality relation: Pxy iff $x = y$, and G would hold whether the universe is finite or infinite. Or P could be the empty relation that doesn't relate any elements at all, in which case G fails in any universe.

The solution is to add clauses to the formula that force the relation P to have the properties we need. Now the irreflexive and transitive properties of relations are enough to guarantee that P is an order in any interpretation. Irreflexivity is $I = (\forall x)[\neg Pxx]$, and transitivity is defined as $T = (\forall x)(\forall y)(\forall z)[Pxy \wedge Pyz \implies Pxz]$.

Together these properties imply that every finite set of elements has a maximal element x such that no y satisfies Pxy. Why? Because if every element had a greater element then by transitivity there would have to be some element x such that Pxx, contradicting irreflexivity. (Think of a directed graph with n vertices. If every vertex has an edge leading out of it, there must be a cycle. This is the same idea.)

But $G = (\forall x)(\exists y)[Pxy]$ implies that there are no P-maximal elements in the universe. So $I \wedge T \wedge G$ is satisfiable in a denumerably infinite universe, for example in $\mathbb{N}$ with $P = <$. But it is not satisfiable in any finite universe under any interpretation because I and T together imply that there is a maximal element in the finite universe, contradicting G.