17 Peano Arithmetic

Is the same as Presburger Arithmetic, but now we add multiplication and its axioms that we could not express before. *See Mendelson chapter 3*

17.1 Axioms of Arithmetic

We add multiplication axioms

1. $S\,x \neq 0$
2. $S\,x = S\,y \Rightarrow x = y$
3. $x + 0 = x$
4. $x + S\,y = S(x + y)$
5. $x * 0 = 0$
6. $x * S\,y = x * y + x)$

Ind: $P[0/z] \wedge (\forall x)(P[x/z] \Rightarrow P[S\,x/z]) \Rightarrow (\forall x)(P[x/z])$ *for every formula P with free variable z*

Q: Do we miss any axioms of the natural numbers?

Commutativity and associativity of multiplication holds as well as the distributive laws for + and *.
Example proof: here is one more of the basic laws: $(\forall x) x * S\,0 = x$. There is a neutral element for multiplication.

$\vdash x * S\,0 = x$ `by RW lhs with Axiom 6`
`1.` $\vdash x * 0 + x = x$ `by RW lhs with Axiom 5`
`1.1.` $\vdash 0 + x = x$ `by RW lhs with commutativity of +`
`1.1.1.` $\vdash x + 0 = x$ `by RW lhs with Axiom 3`
`1.1.1.1.` $\vdash x = x$ `by Equality (reflexivity)`

Note – we didn't even need induction

17.2 Properties

As before the axioms are satisfiable by the standard representation of $< \mathbb{N}, 0, S, +, * >$. Up to isomorphism this is the only model for Peano Arithmetic.

Q: What can we express in Peano Arithmetic?

We can now express all recursive functions. Recall that an n-ary function f can be *represented* in a theory $\mathcal{T}$ if there is an n+1–ary predicate F such that for all $\bar{x}, y$
$f(\bar{x}) = y$ implies $F(\underline{\underline{\bar{x}}}, \underline{\underline{y}})$ is valid and $f(\bar{x}) \neq y$ implies $\neg F(\underline{\underline{\bar{x}}}, \underline{\underline{y}})$ is valid.

1. S, +, *, 0 can obviously be represented
2. We had shown how to represent
   projection $p_i^n(x_1, ..x_n) := x_i$,
   composition h(x) := f(g(x), and
   Minimization, or unrestricted loop or search g(x) := μ_y[f(x,y)=0]

3. What remains is *primitive recursion*

   h(x,0) := f(x), h(x,n+1) := g(x,n,(h(x,n))): g,f represented by G,F

   h(x,y) = t can be represented via unrestricted loops by searching for an the smallest number that correctly encodes the sequence of computations $h(x,0), h(x,1)....h(x,y)$.

   One can encode a sequence $s_1, s_2, .., s_n$ of natural numbers as follows:

   (a) search for the smallest prime number p that is larger than all the s_i

   (b) represent $s_1, s_2, .., s_n$ by the number $\hat{s} = s_0 + s_1{*}p + s_2{*}p^2 \, .. + s_n{*}p^n$

   Both the propterty of being prime and the summation can be expressed using minimization and Formulae from Peano Arithmetic.

   Given $\hat{s}$ we can compute each s_i using minimization s_i: we divide by p^{i-1}, compute the remainder of dividing by p again, and get s_i. Call this computation $\hat{s}(i)$

   The number $\hat{s}$ encoding $h(x,0), h(x,1)....h(x,y)$ is thus $\mu_{\hat{s}}[F(x,\hat{s}(0)) \wedge (\forall n) G(x,n,\hat{s}(i),\hat{s}(i+1))]$

   Once we have computed this number we pick $\hat{s}(y)$ as value for h(x,y).

   So primitive recursion can be represented by minimization if we have addition and multiplication. And since minimization can be represented, we can represent primitive recursion as well.

   See Boolos-Jeffrey chapter 14 for the many details that I skipped.

Now we know from theoretical computer science that all computable functions can be represented by using 0, successor, projections, function composition, primitive recursion, and minimization. So we can express all computable functions in Peano Arithmetic.

$\boxed{\textbf{HOMEWORK:}}$

Represent integer division and a test for being a prime.
SOLUTION
$x \div y = z$ *iff* $D(x,y,z) \equiv (\exists t)(t < y \wedge x = z * y + t)$ *is valid*
$x|y \equiv (\exists z)(x * z = y)$
x prime iff $PR(x,t) \equiv (\forall z)(z|x \Rightarrow (z = 1 \vee z = x)) \Leftrightarrow t = 1$

Q: What about decidability?

Undecidable: mainly because of the recursive functions but there is also a different proof that we shall present later. (via $\mathcal{Q}$ see BJ chapter 15)

Löwenheim Skolem Theorem (BJ chapter 13)

> **Löwenheim-Skolem**: Let S be a set of formulae. If all finite subsets of S are satisfiable, then S is simultaneously satisfiable in a denumerable domain.

That was already discussed in the context of first-order logic (Smullyan page 63). It still holds but becomes more interesting in the context of real numbers.

> **Corollary**: If a formula X that does not contain the equality sign is satisfiable, then it is satisfiable in a model whose domain are the the natural numbers. denumerable domain.

So even if we talk about real numbers – as long as we don't use anything derived from equality, we can give a denumerable model.

Q: Any Problems?

We need infinitely many axioms for induction
This is one of the major drawbacks of both Presburger and Peano Arithmetic. Instead of having a small collection of axioms we need one induction axiom for each arithmetic formula. Our induction rules is in fact a rule scheme or axiom scheme.
The question is, can we axiomatize arithmetic with only finitely many axioms? Or do we lose expressiveness if we replace the induction axioms by some other axiom. This question has led to the study of various induction-free theories of Arithmetic.
If we simply drop the induction axioms, the resulting theory would be of little interest, as one cannot even show that a number that is not zero, must be a successor. So people have studied various axioms that were added to the six basic axioms of Peano Arithmetic. THe most interesting theory is a theory called $\mathcal{Q}$, which we will discuss next.