Blocked Tableau and Sequent Systems are isomorphic

Blocked Tableau		Multi-conclusioned sequents	
left	right	left	right
α $\boldsymbol{S}, \mathsf{T}(A \wedge B)$ $\boldsymbol{S}, \mathsf{T}(A), \mathsf{T}(B)$	$\boldsymbol{S}, \mathsf{F}(A \wedge B)$ β $\boldsymbol{S}, \mathsf{F}(A)$ $\boldsymbol{S}, \mathsf{F}(B)$	`andL` $\boldsymbol{\Delta}, A \wedge B \vdash \boldsymbol{\Gamma}$ $\boldsymbol{\Delta}, A, B \vdash \boldsymbol{\Gamma}$	$\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A \wedge B$ `andR` $\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A$ $\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, B$
β $\boldsymbol{S}, \mathsf{T}(A \vee B)$ $\boldsymbol{S}, \mathsf{T}(A)$ $\boldsymbol{S}, \mathsf{T}(B)$	$\boldsymbol{S}, \mathsf{F}(A \vee B)$ α $\boldsymbol{S}, \mathsf{F}(A)$, $\mathsf{F}(B)$	`orL` $\boldsymbol{\Delta}, A \vee B \vdash \boldsymbol{\Gamma}$ $\boldsymbol{\Delta}, A \vdash \boldsymbol{\Gamma}$ $\boldsymbol{\Delta}, B \vdash \boldsymbol{\Gamma}$	$\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A \vee B$ `orR` $\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A, B$
β $\boldsymbol{S}, \mathsf{T}(A \supset B)$ $\boldsymbol{S}, \mathsf{F}(A)$ $\boldsymbol{S}, \mathsf{T}(B)$	$\boldsymbol{S}, \mathsf{F}(A \supset B)$ α $\boldsymbol{S}, \mathsf{T}(A), \mathsf{F}(B)$	`impL` $\boldsymbol{\Delta}, A \Rightarrow B \vdash \boldsymbol{\Gamma}$ $\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A$ $\boldsymbol{\Delta}, B \vdash \boldsymbol{\Gamma}$	$\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A \Rightarrow B$ `impR` $\boldsymbol{\Delta}, A \vdash \boldsymbol{\Gamma}, B$
α $\boldsymbol{S}, \mathsf{T}(\sim A)$ $\boldsymbol{S}, \mathsf{F}(A)$	$\boldsymbol{S}, \mathsf{F}(\sim A)$ α $\boldsymbol{S}, \mathsf{T}(A)$	`notL` $\boldsymbol{\Delta}, \neg A \vdash \boldsymbol{\Gamma}$ $\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, A$	$\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}, \neg A$ `notR` $\boldsymbol{\Delta}, A \vdash \boldsymbol{\Gamma}$
$*$ $\boldsymbol{S}, \mathsf{T}(A), \mathsf{F}(A)$		`axiom` $\boldsymbol{\Delta}, A \vdash \boldsymbol{\Gamma}, A$	

Differences between Tableau and Sequent Systems

	Tableau	Sequent
Goal	$\mathsf{F}(X)$: search for counterexample	$\vdash X$: try to prove X
α-step	counterexample must involve both α_1 and α_2 (AND branch)	proof must show either α_1 or α_2 (OR-branch)
β-step	counterexample uses either β_1 or β_2 (OR-branch)	proof must show both β_1 and β_2 (AND branch)
closed branch	counterexample impossible	partial proof sucessful
open leaf	consistent valuation possible (counterexample found)	proof fails at this leaf
	implicit proof of validity – no counterexample	explicit justification of validity
	negative / indirect approach	positive / constructive proof

What is wrong with multiple conclusions?

Pierce's Law: $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Top :	$\vdash ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$	by `impR`
Top 1 :	$(p \Rightarrow q) \Rightarrow p \vdash p$	by `impL`
Top 1 1 :	$\vdash p,\ p \Rightarrow q$	by `impR`
Top 1 1 1:	$p \vdash p,\ q$	by `axiom`
Top 1 2 :	$p \vdash p$	by `axiom`

We switched the proof goal in [Top 1 1 1] !!!

Construction of proof argument not visible

$\Downarrow$

Restrict calculus to single conclusion

Multi- and Single-Conclusioned Sequent Systems

Multi-conclusioned sequents	
left	right
$\mathbf{\Delta}, A \wedge B \vdash \mathbf{\Gamma}$ $\mathbf{\Delta}, A, B \vdash \mathbf{\Gamma}$	$\mathbf{\Delta} \vdash \mathbf{\Gamma}, A \wedge B$ $\mathbf{\Delta} \vdash \mathbf{\Gamma}, A$ $\mathbf{\Delta} \vdash \mathbf{\Gamma}, B$
$\mathbf{\Delta}, A \vee B \vdash \mathbf{\Gamma}$ $\mathbf{\Delta}, A \vdash \mathbf{\Gamma}$ $\mathbf{\Delta}, B \vdash \mathbf{\Gamma}$	$\mathbf{\Delta} \vdash \mathbf{\Gamma}, A \vee B$ $\mathbf{\Delta} \vdash \mathbf{\Gamma}, A, B$
$\mathbf{\Delta}, A \Rightarrow B \vdash \mathbf{\Gamma}$ $\mathbf{\Delta} \vdash \mathbf{\Gamma}, A$ $\mathbf{\Delta}, B \vdash \mathbf{\Gamma}$	$\mathbf{\Delta} \vdash \mathbf{\Gamma}, A \Rightarrow B$ $\mathbf{\Delta}, A \vdash \mathbf{\Gamma}, B$
$\mathbf{\Delta}, \neg A \vdash \mathbf{\Gamma}$ $\mathbf{\Delta} \vdash \mathbf{\Gamma}, A$	$\mathbf{\Delta} \vdash \mathbf{\Gamma}, \neg A$ $\mathbf{\Delta}, A \vdash \mathbf{\Gamma}$
$\mathbf{\Delta}, A \vdash \mathbf{\Gamma}, A$	

Single-conclusioned sequents			
	left	right	
`andL`	$\mathbf{\Delta}, A \wedge B \vdash \boldsymbol{G}$ $\mathbf{\Delta}, A, B \vdash \boldsymbol{G}$	$\mathbf{\Delta} \vdash A \wedge B$ $\mathbf{\Delta} \vdash A$ $\mathbf{\Delta} \vdash B$	`andR`

`orL`	$\mathbf{\Delta}, A \vee B \vdash \boldsymbol{G}$
	$\mathbf{\Delta}, A \vdash \boldsymbol{G}$
	$\mathbf{\Delta}, B \vdash \boldsymbol{G}$
		$\mathbf{\Delta} \vdash A \vee B$
		$\mathbf{\Delta} \vdash A$
		
		$\mathbf{\Delta} \vdash A \vee B$
		$\mathbf{\Delta} \vdash B$
			`orR1`
			
			`orR2`

`impL`	$\mathbf{\Delta}, A \Rightarrow B \vdash \boldsymbol{G}$ $\mathbf{\Delta} \vdash A$ $\mathbf{\Delta}, B \vdash \boldsymbol{G}$	$\mathbf{\Delta} \vdash A \Rightarrow B$ $\mathbf{\Delta}, A \vdash B$	`impR`
`notL`	$\mathbf{\Delta}, \neg A \vdash \boldsymbol{G}$ $\mathbf{\Delta} \vdash A$	$\mathbf{\Delta} \vdash \neg A$ $\mathbf{\Delta}, A \vdash$ `false`	`notR`
`axiom`	$\mathbf{\Delta}, A \vdash A$		

Did we lose anything?

- **Pierce's Law:** $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Top :	$\vdash ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$	by `impR`
Top 1 :	$(p \Rightarrow q) \Rightarrow p \vdash p$	by `impL`
Top 1 1 :	$\vdash p \Rightarrow q$	by `impR`
Top 1 1 1:	$p \vdash q$	by `???`
Top 1 2 :	$p \vdash p$	by `axiom`

We cannot complete the proof in [Top 1 1 1]

- **But Pierce's law is true**
  - There are multi-conclusioned sequent proofs for it
  - There is a truth table proof for $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Add truth-table assumptions to sequent proof

As first step, assert (`cut`) that p is true or p is false

Top : $\vdash ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$	by `cut` $p \vee \neg p$
Top 1 : $\vdash p \vee \neg p$	by `magic`
Top 2 : $p \vee \neg p \vdash ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$	by `impR`
Top 2 1 : $p \vee \neg p,\ (p \Rightarrow q) \Rightarrow p \vdash p$	by `orL`
Top 2 1 1 : $p,\ (p \Rightarrow q) \Rightarrow p \vdash p$	by `axiom`
Top 2 1 2 : $\neg p,\ (p \Rightarrow q) \Rightarrow p \vdash p$	by `impL`
Top 2 1 2 1 : $\neg p, \vdash p \Rightarrow q$	by `impR`
Top 2 1 2 1 1 : $\neg p,\ p \vdash q$	by `notL`
Top 2 1 2 1 1 1 : $p \vdash p$	by `axiom`
Top 2 1 2 2 : $\neg p,\ p \vdash p$	by `axiom`

`magic` rule is disputed

Multi- vs Single-Conclusioned Sequent Systems

Single-Conclusioned Sequent Systems + `magic` + `cut`

≡

Multi-Conclusioned Sequent Systems

Cut-Elimination Theorem

Single-Conclusioned ≡ Single-Conclusioned + `cut`

Single-Conclusioned + `magic` ≡ Single-Conclusioned + `magic` + `cut`

- **Constructive Logic ≡ Single-Conclusioned Calculus**
- **Classical Logic ≡ Single-Conclusioned + `magic`**
- **Refinement Logic: computer-ready version**

  Using *lists* of assumptions instead of sets

Refinement Logic

left		right	
andL i	$\boldsymbol{H}, A \wedge B, \boldsymbol{H'} \vdash G$ $\boldsymbol{H}, A, B, \boldsymbol{H'} \vdash G$	$\boldsymbol{H} \vdash A \wedge B$ $\boldsymbol{H} \vdash A$ $\boldsymbol{H} \vdash B$	andR
orL i	$\boldsymbol{H}, A \vee B, \boldsymbol{H'} \vdash G$ $\boldsymbol{H}, A, \boldsymbol{H'} \vdash G$ $\boldsymbol{H}, B, \boldsymbol{H'} \vdash G$	$\boldsymbol{H} \vdash A \vee B$ $\boldsymbol{H} \vdash A$ $\boldsymbol{H} \vdash A \vee B$ $\boldsymbol{H} \vdash B$	orR1 orR2
impL i	$\boldsymbol{H}, A \Rightarrow B, \boldsymbol{H'} \vdash G$ $\boldsymbol{H}, A \Rightarrow B, \boldsymbol{H'} \vdash A$ $\boldsymbol{H}, \boldsymbol{H'}, B \vdash G$	$\boldsymbol{H} \vdash A \Rightarrow B$ $\boldsymbol{H}, A \vdash B$	impR
notL i	$\boldsymbol{H}, \neg A, \boldsymbol{H'} \vdash G$ $\boldsymbol{H}, \neg A, \boldsymbol{H'} \vdash A$	$\boldsymbol{H} \vdash \neg A$ $\boldsymbol{H}, A \vdash \texttt{false}$	notR
hypothesis i	$\boldsymbol{H}, A, \boldsymbol{H'} \vdash A$		

Refinement Logic – Special Rules

	$\boldsymbol{H} \vdash \mathrm{G}$ `cut` $\boldsymbol{A}$ $\boldsymbol{H} \vdash \boldsymbol{A}$ $\boldsymbol{H},\ \boldsymbol{A} \vdash \mathrm{G}$
`magic` $\boldsymbol{H} \vdash \mathrm{A} \vee \neg\mathrm{A}$	$\boldsymbol{H} \vdash \mathrm{G}$ `ClassCases` $\boldsymbol{A}$ $\boldsymbol{H},\ \boldsymbol{A} \vdash \mathrm{G}$ $\boldsymbol{H},\ \neg\boldsymbol{A} \vdash \mathrm{G}$
`lemma` *name* $\boldsymbol{H} \vdash \mathrm{L}$	$\boldsymbol{H} \vdash \mathrm{G}$ `cut_lemma` *name* $\boldsymbol{H},\ \mathrm{L} \vdash \mathrm{G}$

Exercises

- Prove the formulae in Smullyan, p24, Exercise 6,7,8 using Refinement Logic
- (Extra): Can the formulae in Smullyan, p24, Exercise 5 be shown without the `magic` rule?
- Show that the rules `ClassCases` and `cut_lemma` are *redundant* in Refinement Logic:
  show how to transform a proof that uses these rules into one that does not use them.

Please submit each homework problem on a separate sheet of paper