Topic: COMPACTNESS Smullyan p.30–36

Assignments Exercises: Smullyan p.36 — Reading: Smullyan p.101–108

5 Compactness

So far we have applied the tableau method to single formulae and have shown that this method is works well to show that a formula is *valid*, i.e. a tautology. The tableau method is a finite proof method and can be implemented on a computer. Now the question is – where are the limitations of such finite methods. Can we use them to investigate sets of formulae, even if they are infinite? Will they be useful to get an interesting results? Can we use finite methods to something infinite? This question is not only mathematically interesting but will also tell us to what extent we can use the method for *first order logic*, when quantifiers will turn a single formula into a description of infinite sets of formulae.
In mathematics, this question relates to the subject of *compactness*: given a possibly infinite set S of formulae – what can we say about it when we know the properties of its finite subsets? In particular, what can we say about its *satisfiability*, i.e. whether there is a valuation $v_0 : Var \rightarrow \mathbb{B}$ which satisfies S ($val(X, v_0) = t$ for all $X \in S$)?
We know that we can use the tableau method to decide whether a formula X is satisfiable (if a complete tableau for TX is open) or not (if the tableau is closed). We can use the same method to decide whether a *finite* conjunction of formulae $x_1 \wedge X_2 \ldots X_n$ is satisfiable. We also know that an infinite set S of formulae is not satisfiable, if a finite subset of it is already unsatisfiable. But can we be sure that S is satisfiable if every finite subset of S has an open tableau?
Well, this obviously does not depend on the method of tableau but only on the question whether the satisfiability of all finite subsets implies the satisfiability of S itself. Well, the answer is YES. It would be very unfortunate if it were not so, because finite methods like the tableau method wouldn't help us very much in first order logic in this case.
The interesting question is, however, how to prove this fact. And because the ways in which this compactness theorem is proven are so fundamental for the development and study of proof methods, Smullyan gives us a variety of different proofs for it. But first let us state the theorem. We define a set S of formula to be *consistent*, if every finite subset of S is satisfiable. The theorem thus can be formulated as

Theorem 5.1 (Compactness Theorem)
Every infinite consistent set of formulae is satisfiable

5.1 Analytic proofs

The first kind of proofs for this theorem will be based on tableau and a related, more general problem from mathematics, namely the question whether every tree T that has only finite branches, is necessarily finite itself, or – to be more precise – has a uniform length limit for all branches, that is a maximal level. Formally: ($\forall$b:Branch(T).$\exists$L:$\mathbb{N}$. lg(b)$\leq$L $\supset$ $\exists$L:$\mathbb{N}$. $\forall$b:Branch(T).lg(b)$\leq$L)?
Q: *any suggestions*
Well, the answer depends. If we allow a point to have infinitely many successors, then the answer is obviously NO (*illustrate why*).
But if we consider *finitely generated trees* (and formulae and tableau are of that kind) then there is in fact a uniform length limit. This insight is called

Lemma 5.1 (König's Lemma)
Every finitely generated tree that has only finite branches is finite

To prove this lemma, let T be a finitely generated tree and assume it has infinitely many points. We will construct an infinite branch in T by considering points that we call *good*. A point is good, if it has infinitely many descendants.
Obviously the origin a_1 is good. Now, if a point is good and has only finitely many successors, one of them must be good as well. Using this argument we recursively build an infinite chain $a_1, a_2, a_3, \ldots$ of points that form a branch in T. Q: *why does this prove the lemma?*

Using this lemma we can prove the compactness theorem as follows:
Assume that each subset $\{X_1, ...X_n\}$ of S is satisfiable. Then each tableau proof for $X_1 \wedge ...X_n$ must have open branches. We construct a tableau tree for S as follows:
Construct a complete tableau for X_1. Add X_2 to each open branch of the tableau and $X_1 \wedge X_2$ at the top. Continue the development of the tableau for $X_1 \wedge X_2$ until it is complete. Add X_3 to the open branches and continue. Since each of the tableau has open branches this procedure contructs an infinite tree that contains all $X_i \in S$. Because of König's Lemma, this tree must have an infinite branch. By construction, all the X_i are on this branch and the set S^* of all formulae on this branch is a Hintikka set (that is due to the tableau method). By Hintikka's lemma, which we had just a while ago, the setS^* must be satisfiable and so is S as a subset of S^*.

This proof not only shows compactness but also explains how we could define a tableau for an infinite set of formulae: We start with a formula $X \in S$ and apply in each step either one of the tableau rules A and B or add an arbitrary formula from S to the end of the branch. If the tableau is not closed, it will become infinite and thus show the satisfiability of S.

Smullyan also gives another proof for the compactness theorem that directly constructs a Hintikka set without using König's lemma or the tableau method. If you look at this proof on pages 34/35 you may get the impression that this is almost the same proof as before, but it gets to the same result with less overhead.
Exercise: Assume for every valuation v_i there is an $X_i \in S$ with $val(X_i, v_i) = t$. Show by using the compactness theorem that for some n the disjunction $X_1 \vee ...X_n$ is a tautology.

5.2 Lindenbaum's theorem

The above proofs used Hintikka's lemma to show the satisfiability of the set S. There is a different proof, that shows us how to build the validation that satisfies the set S. This proof is due to Lindenbaum and very useful for our study of first order logic. The basic idea is to extend the set S into a maximally consistent set and to extract the validation from that set.
A set S is *maximally consistent* if it is consistent and no proper superset is consistent. We will first show how to extract a validation from a maximally consistent set.

Lemma 5.2 *Every maximally consistent set S is a truth set, that is*
$\exists v_0 : Var_S \to \mathbb{B}. S = \{X | val(X, v_0) = t\}$.

This is easy to see if we look at the conditions for truth sets on page 12 (or 23). If S is maximally consistent, then for each formula X either X or $\sim X$ must belong to S (one of them is consistent with S). If X and Y belong to S, then $X \wedge Y$ is consistent with S and must belong to S. If $X \wedge Y$ belongs to S then X and Y are consistent with S and must belong to it. Similar we prove the conditions S_2 and S_3 on page 12. (Smullyan actually refers to a derived condition on page 23).

Now if S is a truth set, then we only have to look at the propositional variables. We define $v_0(p) := t$ if $p \in S$ and $v_0(p) := f$ if $\sim p \in S$. By construction v_0 is the valuation that satisfies S.

Now the crucial idea part of Lindenbaum's proof is that we can extend every consistent set into a maximally consistent one – roughly by adding consistent formulae to S until there is no more left. Obviously this is an infinite process but as long as we have only denumerably many formulae, we can describe this process recursively and also give a precise description of the resulting maximally consistent set.
Actually, for this proof we only need the fact that consistency is a property that has a *finite character* :
F: a set S is consistent iff all finite subsets of S are consistent
This fact is easy to see. If S is consistent, then all finite subsets of its finite subsets are satisfiable, so its finite subsets are consistent. On the other hand if all finite subsets are consistent, then the are satisfiable (because they are finite subsets of themselves) and this S is consistent.
More generally we say that property P has finite character if $P(S)$ holds if and only if $P(S')$ holds for all finite subsets of S. For such properties each set S that has property P can be extended into a maximal set that has property P. For denumarable universes, this fact is called

Lemma 5.3 (Tukeys Lemma)
Let $U = \{Y_1, Y_2, ...\}$ be a denumerable universe, P be a property on subsets of U of finite character, and $S \subseteq U$ with $P(S)$. Then there is a maximal set $S^* \subseteq U$ with $P(S^*)$ and $S \subseteq S^*$.

For the proof we construct sets S_i inductively as follows: $S_0 := S$, $S_{n+1} := S_n \cup \{Y_{n+1}\}$ if this set is consistent and $S_{n+1} := S_n$ otherwise. We define $S^* := \bigcup S_i$. Then
$P(S^*)$ holds: If $K \subseteq S^*$ is finite then $K \subseteq S_i$ for some i. We know $P(S_i)$ by construction, and because of the finite character also $P(K)$ and also $P(S^*)$.
Maximality: if $P(S^* \cup \{Y_i\})$ holds then also $P(S_{i-1} \cup \{Y_i\})$. Thus $Y_i \in S_i \subseteq S^*$.

Because the set of all formulae is denumerable we get from this

Theorem 5.2 (Lindenbaum's Theorem)
Every consistent set of formulae can be extended into a maximally consistent one.

Lindenbaum's construction can be modified into an analytic one that does not consider all possible formulae but – as the constructions we used earlier – only those who describe subformulae of elements of S. Formally these formulae are defined by the *descendants* of of formula X in S, where Y is a *direct descendent* of X if X is α or β and Y is α_1,α_2, β_1, or β_2.
Now consider $S^0 := \{Y | \exists X \in S. Y \text{ descendant of } X\}$, which is obviously a superset of S. By Lindenabaum's construction we know that S has a maximally consistent extension S^* in the universe S^0. Because of the limitation to S^0 this set is not necessarily a truth set. But one can easiliy prove that

Theorem 5.3 *For each consistent set S every maximally consistent subset of S^0 is a Hintikka set.*

We only have to prove the conditions H_0, H_1, H_2 for Hintikka sets which follow more or less directly from the maximality. (p26)

5.3 The compactness theorem for deducibility

There is one, quite useful form of the compactness theorem which has to do with the notion of deducing a fact from certain assumptions. We say that X *is deducible from S* if for some

$X_i \in S$ the formula $X_1 \wedge .. X_n \supset X$ is a tautology. Now the compactness theorem relates formal deducibility to the notion of *logical consequence*.

Theorem 5.4 (Compactness / Deduction Theorem)
If X is true in all valuations that satisfy S then X is deducible from S.

Since $S \cup \{\sim X\}$ is unsatisfiable there must be a finite subset $S_0 = \{X_1, ..., X_n, \sim X\}$ of $S \cup \{\sim X\}$, which is also unsatisfiable (compactness, not assuming S to be consistent). Thus $X_1 \wedge .. X_n \supset X$ must be a tautology.

O.k – so far for the theoretical foundations of propositional logic and tableau calculi. On Thursday we want to discuss alternative calculi which are a bit better for computerized proofs – either by being more compact to allow more automated proof search, or by emphasizing the concept of logical deduction to support interaction.