Page 30 #3. Show that the the two tableaux rules for $|$ alone give a complete and consistent proof system for propositional logic. The result may be intuitive but a rigorous proof relating the $|$ connective and the nature of the tableau proof system was required for full credit. More specifically, that means showing:

**Consistency**: If you can prove a formula $X$ using a tableau with only the two $|$ rules, then $X$ must be a tautology. That is, if there exists a tableau starting with $F\,X$ that is extended by those two rules and, when completed, has all closed paths, then $X$ must be a tautology (because there is no satisfying assignment of values to variables for the formula $F\,X$). So this $|$-tableau method can prove only tautologies.

One way to prove consistency is to follow the model on page 25. Prove by induction that any extension of a tableau $\mathcal{T}$ which is true is again true, provided that the extension is exactly one that results from one of the $|$ rules. Such a proof would mention extending a true tableau $\mathcal{T}_1$ to $\mathcal{T}_2$ along a true branch $\theta$ and would explain why both of the $|$ rules preserve the truth of a path. Then for any tableau $\mathcal{T}$ if the origin $F$ is true under a given interpretation $v_0$ (an interpretation assigns truth values to variables), then $\mathcal{T}$ must be true under $v_0$. A closed tableau cannot be true under any interpretation, so the origin $F$ of a closed tableau must be unsatisfiable, so the negation of the origin is a tautology. So a tableau proof of $F$ (a closed, completed tableau with $\sim F$ at the origin extended by the two $|$ rules) implies that $F$ is a tautology. So this tableau method is consistent.

**Completeness**: The object here is to show that if a formula $X$ is a tautology (that uses only the $|$ connective) then it is provable via a tableau using only those two $|$ rules. So for any such tautology $X$ any completed tableau starting with $F\,X$, extended by the two $|$ rules, must have all closed paths.

Smullyan's chain of reasoning on pages 26-29 can be used almost unaltered to show the completeness of this new tableau system. In order for that proof to work, however, that the a new proof for Hintikka's Lemma must be given in terms of the explicit $|$-rules (so that it holds for formulae using only the $|$ connective). That is, you must show that for any open branch $B$ in any completed $|$-tableau $\mathcal{T}$, the elements of the branch are a Hintikka set and are thus satisfiable. So if $X$ is a tautology, then every completed $|$-tableau starting with $F\,X$ (which is not satisfiable) must have all closed branches.

Optionally, you can solve the exercise on page 29 which proves the completeness theorem without using Hintikka's Lemma. In this case you would replace the generic $\alpha$ and $\beta$ rules Smullyan mentions with the appropriate specific $|$ rules.

**Common misconceptions**.

"Can I just show that any $|$-formula is equivalent to a standard ($\vee$, $\wedge$, etc) formula? Then since the standard tableau method is complete and consistent, the $|$-tableau method is as well?" Unfortunately, there is more involved. If $F$ is a $|$-formula and a tautology, then you can convert it into an equivalent standard formula (which will also be a tautology) and then prove it using a standard tableau. But that doesn't mean that a $|$-tableau can prove all tautologies (completeness). And it doesn't mean that a $|$-tableau proof of a formula means the formula is a tautology (consistency). You could instead try to prove (perhaps by induction on the height of the tableau) that every $|$-tableau for $F$ is equivalent to a unique standard tableau for an equivalent standard formula $F'$, and that the one is closed if and only if the other is as well. Then any $F$ would be $|$-tableau provable iff it were standard tableau provable, which would imply the consistency and completeness of the $|$-tableau method. But that's probably a longer proof than the "extend a true tableau" proof. In particular, this problem does not follow immediately from page 14, exercise #5.

"No, really. Can't I just relate $(p|q) \Leftrightarrow (\sim (p \wedge q))$ from page 14, #5 and then rederive the new rules in terms of the old:

| $\mathrm{T} \sim (X \wedge Y)$ | $\mathrm{F} \sim (X \wedge Y)$ |
|---|---|
| $\mathrm{F}\ (X \wedge Y)$ | $\mathrm{T}\ (X \wedge Y)$ |
| $\mathrm{F}\ X \mid \mathrm{F}\ Y$ | $\mathrm{T}\ X$ |
| | $\mathrm{T}\ Y$ |

And then it must be correct and consistent?" No. It might be possible to argue consistency in this fashion, since if you remove any of the standard tableau rules the system remains consistent (why? see page 26). So a slightly stronger argument showing that a tableau with the two $|$-rules is equivalent to a standard tableau with a subset of the standard rules (perhaps proven inductively on the height of the tableau) would be a consistency argument. A subset of the standard tableau rules need not be complete, however (see page 26 again). So you must still explain why we can prove all tautologies in the (apparently weaker) $|$-tableau system.

"Because the $|$ connective can be used to define all of the other connectives, the $|$-tableau rules are complete." Well, yes, but there is more involved in a complete proof of this. In particular, you want to show that if a $|$-formula $X$ is a tautology then any complete $|$-tableau of $F\,X$ must have all closed branches (since this is the definition of a tableau proof of $X$). Intuitively, the $|$ connective (and its associated tableau method) is certainly strong enough to prove all true theorems. But to make your proof rigorous you must match it with the technical definition of completeness.

"Because we can derive the 8 standard tableau rules from the $|$ rules, and the standard tableau rules are correct, the $|$-tableau system must be correct." This is false. Do not confuse completeness and correctness. Completeness is (loosely) about being able to prove "enough" things (namely, all the tautologies). Correctness (consistency) is about not being able to prove "too many" things (namely, any non-tautology). Just because we can prove at least as many things as the standard system (by simulating the standard connectives and rules) does not mean that we cannot prove more things (perhaps the $|$ connective is too powerful and allows you to prove anything). In particular, consistency is preserved when we take away tableau rules (if we can prove something with only a few good rules, it is still true), and completeness is preserved when we add additional (possibly "bad") rules (if we could prove all the true things before, we can certainly still prove them with more tools at our disposal). To prove consistency you must show that a $|$-tableau proof of $X$ (that is, completed $|$-tableau for $F\,X$ with all closed paths) implies that $X$ is a tautology.

"Can't I just prove that the $|$-rules are correct?" The question wants you to prove that the rules give a complete and correct *system*, not merely that the two $|$-rules are consistent. Having consistent rules is a necessary condition for a consistent system, but showing that the rules make sense logically does not imply (for example) that you can prove all tautologies (why?).

"This question has only two parts." Sadly, no (see page 30). You were also supposed to include tableau rules for $\downarrow$. This part was commonly overlooked, but it was only worth five points.

Please turn in all of your homework problems on separate sheets of paper (at most one problem per sheet).