CS 486

Lecture 6: Handout on Matrix Methods

type α successor polarities	$(A \wedge B)^T$ $A^T,\ B^T$	$(A \vee B)^F$ $A^F,\ B^F$	$(A \supset B)^F$ $A^T,\ B^F$	$(\sim A)^T$ A^F	$(\sim A)^F$ A^T
type β successor polarities	$(A \wedge B)^F$ $A^F,\ B^F$	$(A \vee B)^T$ $A^T,\ B^T$	$(A \supset B)^T$ $A^F,\ B^T$		

Constructing the Matrix of a formula X

1. Starting with X^F assign types and polarities to each subformula of X according to the above table.

   $(p \vee (q \wedge r)) \supset ((p \vee q) \wedge (p \vee r))$ becomes $(p^T \vee_\beta^T (q^T \wedge_\alpha^T r^T)) \supset_\alpha^F ((p^F \vee_\alpha^F q^F) \wedge_\beta^F (p^F \vee_\alpha^F r^F))$

   For simplicity, remove polarities from connectives, replace F by an overbar and remove T

   $(p \vee_\beta (q \wedge_\alpha r)) \supset_\alpha ((\bar{p} \vee_\alpha \bar{q}) \wedge_\beta (\bar{p} \vee_\alpha \bar{r}))$

2. (Signed) formulae p and q *α-related* if the last connective that links p and q is of type α. Otherwise they are *β-related*

3. Write X in a two-dimensional fashion as a *matrix*, placing α-related atoms side by side and β-related ones on top of each other. To preserve the tree structure of the formula, we put brackets around the representation of each subformula. As a convention, we read the two-dimensional representation of the matrix as follows: if $\mathcal{M} \neq \mathcal{M}'$ are the largest (visible) sub-matrices that contain the atoms p and q, then p and q are α-related if $\mathcal{M}$ is left or right of $\mathcal{M}'$ and β-related if $\mathcal{M}$ is above or below $\mathcal{M}'$. Brackets can be omitted, as long as their omission does not change the interpretation of the matrix.

   $$\left[\begin{array}{ccc} [\,q & r\,] & [\,\bar{p} \quad \bar{q}\,] \\ & p & [\,\bar{p} \quad \bar{r}\,] \end{array}\right] \qquad\qquad \left[\begin{array}{ccc} [\,q & r\,] & [\,\bar{p} \quad \bar{q}\,] \\ & p & [\,\bar{p} \quad \bar{r}\,] \end{array}\right]$$

   In a matrix a tableau *path* corresponds to a horizontal path from left to right.

4. A *complementary* connection is a pair of atoms with the same variable and different polarities.

   $$\left[\begin{array}{ccc} [\,q & r\,] & [\,\bar{p} \quad \bar{q}\,] \\ & p & [\,\bar{p} \quad \bar{r}\,] \end{array}\right] \qquad\qquad \left[\begin{array}{ccc} [\,q & r\,] & [\,\bar{p} \quad \bar{q}\,] \\ & p & [\,\bar{p} \quad \bar{r}\,] \end{array}\right]$$

   A path is closed if it contains a complementary connection. All 4 paths are closed.

Theorem: *A propositional formula X is a tautology iff every path through a matrix representation of X contains a complementary connection.*

Proof Methodology: To check that all paths are closed, follow connections first.

Further Details:

- W. Bibel. On matrices with connections. *Journal of the ACM*, 28:633–645, 1981.
- C. Kreitz & J. Otten. Matrix-based Theorem Proving in Classical and Non-classical Logics, *Journal for Universal Computer Science, Special Issue on Integration of Deduction Systems*, Springer-Verlag, 1999.

When to use brackets in a two-dimensional matrix-representation

A detailed representation of a formula as matrix would have to use brackets in the two-dimensional representation wherever the formula contains a parenthesis.
The formula $(\bar{p} \vee_\beta \bar{q}) \wedge_\alpha (p \vee_\beta q)$ would be represented as

$$\begin{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} & \begin{bmatrix} p \\ q \end{bmatrix} \end{bmatrix}$$

Even if we omit the brackets it remains obvious that $\bar{p}$ and $\bar{q}$ are β-related (and so are p and q) because they are on top of each other, while $\bar{p}$ and p (and $\bar{p}$ and q, $\bar{q}$ and p, $\bar{q}$ and q) are α-related because they are side by side in the two-dimensional representation. It is therefore safe to omit the brackets and use the following matrix.

$$\begin{bmatrix} \bar{p} & p \\ \bar{q} & q \end{bmatrix}$$

The formula $(\bar{p} \wedge_\alpha p) \vee_\beta (\bar{q} \wedge_\alpha q)$ would be represented as

$$\begin{bmatrix} \begin{bmatrix} \bar{p} & p \end{bmatrix} \\ \begin{bmatrix} \bar{q} & q \end{bmatrix} \end{bmatrix}$$

As before $\bar{p}$ and $\bar{q}$ are β-related and so are p and q. $\bar{p}$ and p are α-related and so are $\bar{q}$ and q. But $\bar{p}$ and q or $\bar{q}$ and p are *not* α-related but β-related. Omitting the brackets would make us assume that $\bar{p}$ is left of q and thus α-related. Since this is not the case, the brackets must remain.

Our example formula $(p \vee_\beta (q \wedge_\alpha r)) \supset_\alpha ((\bar{p} \vee_\alpha \bar{q}) \wedge_\beta (\bar{p} \vee_\alpha \bar{r}))$ from the previous page would be represented as

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} q & r \end{bmatrix} \\ p \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} \bar{p} & \bar{q} \end{bmatrix} \\ \begin{bmatrix} \bar{p} & \bar{r} \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

The brackets denoting the immediate subformulae of the formula do not provide any additional information and can be dropped.

$$\begin{bmatrix} \begin{bmatrix} q & r \end{bmatrix} & \begin{bmatrix} \bar{p} & \bar{q} \end{bmatrix} \\ p & \begin{bmatrix} \bar{p} & \bar{r} \end{bmatrix} \end{bmatrix}$$

Of course one could always leave the brackets in the matrix. But as with many parentheses in a formula, it is somewhat awkward to do so and too much to write anyway.

As a general rule one might say that being β-related binds stronger than being α-related and that both relations are associative. So subformulae of an α-connective do not need brackets but α-subformulae of a β-connective must be denoted by brackets.