CS486: Spring 1999

# Problem Set 7:

Due Thursday, March 18, 1999

## Reading:

Please read Smullyan, pp. 70–78.

## Exercises

*Please submit each homework problem on a separate piece of paper.*

1. Give a Refinement Logic proof of $\vdash \sim (\forall x Px) \supset (\exists x \sim Px)$

2. Monoids

   A monoid is an algebraic structure with one operation, $*$, which is *associative* and has a unit element, 1, such that $x * 1 = x$ and $1 * x = x$.

   (a) Write first order axioms to characterize a monoid. Use the relation *Op(x, y, z)* for $x * y = z$. Use the relation *Unit(x)* to say that $x$ is the unit, and if necessary use $E(x, y)$ for equality.

   Say that a monoid is *commutative* iff $x * y = y * x$, and is *idempotent* iff $x * x = x$.

   (b) Write the axioms for a commutative idempotent monoid.

   Prove or disprove using first-order logic that:

   (a) The unit element of a monoid is unique.

   (b) An idempotent monoid is commutative.

3. Boolean Rings

   (a) A Boolean ring $R$ is a ring (with unit) such that $x * x = x$ for all $x \in R$. Show that any Boolean ring is commutative, i.e., $x * y = y * x$ for all $x, y \in R$.

(b) Define $x + y$ as $x \Leftrightarrow y$, $-x$ as $x$, and $x * y$ as $x \vee y$.
Take 0 as true (T) and 1 as false ($\perp$).

Show that with these definitions the class of propositions, *Prop*, is a Boolean ring. Use the truth definition to prove properties of the logical connectives and constants.

Note, you need not express this in first-order logic.