CS 482 Summer 2005

Homework Assignment #8

Out: August 10 Due: August 16

Question 1

A *dominating set* on a graph G is a subset of nodes S such that every node v is either in S or is adjacent to at least one node in S. The DOMINATING SET (DS) problem takes as input a graph $G = (V, E)$ and an integer k, and determines whether G has a dominating set of size at most k.

(a) Give an example of a graph whose smallest dominating set is different from its smallest vertex cover.

(b) Show that DOMINATING SET is NP-complete. (Hint: Try a reduction from 3SAT, using 3 nodes per variable and one node per clause.)

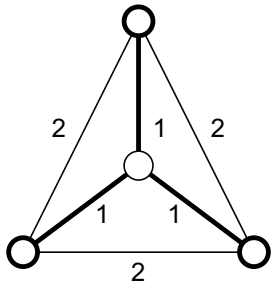
(c) Part (b) shows that we don't know of any polytime algorithm to solve dominating set. However, if our graph is a tree, then this is no longer the case. Give a polytime greedy algorithm that takes in a tree $T = (V, E)$ and returns a dominating set of maximum size.

(d) Now suppose that every node i has a weight w_i. Give a polytime algorithm that takes in a tree $T = (V, E)$ and weight w_v for each node $v \in V$, and returns the dominating set of minimum total weight.

Question 2

Suppose you are given a graph $G = (V, E)$, with costs c_e for each edge e, and a set $V' \subseteq V$ of *terminal nodes*. Your goal is to find a minimum-cost tree $T \subseteq G$ that spans the vertices of V'. While T must include all the nodes of V', in building T you can choose to either use or not use any of the other nodes in V. The cost of T is just the sum of the costs of the edges in T. As shown in the example below, including additional nodes can sometimes actually yield a cheaper tree. Any vertex in T but not in V' is called a *Steiner node*, and T itself is called a *Steiner tree*. The problem of determining whether there is a Steiner Tree of cost $\leq C$ is known as the *STEINER MINIMUM TREE* problem. In the example below, the terminals are the three bold nodes, and the min cost Steiner tree includes the middle non-terminal, or Steiner node.

(a) Show that SMT is NP-Hard, using a reduction from SET COVER.



(b) Suppose the set of terminals V' is only two nodes, namely t_1 and t_2. Show that SMT is solvable in polytime by giving an algorithm that solves it.

(c) Suppose the set of terminals V' is only three nodes, namely t_1, t_2, and t_3. Show that SMT is solvable in polytime by giving an algorithm that solves it.

(c) Suppose the set of terminals V' is all the vertices (i.e. $V' = V$). Show that SMT is solvable in polytime by giving an algorithm that solves it.

(d) Suppose our set of vertices V are the points in the Euclidean plane. In particular, the vertices of V obey the triangle inequality: $d(x, y) + d(y, z) \leq d(x, z)\ \forall x, y, z \in \Re^2$, where d is just the Euclidean distance function. Let the edge (x, y) have cost $d(x, y)$, for any two points x and y in the plane. Now suppose you are given $V' \subseteq V$, a set of points you want to connect using as cheap a tree as possible. Consider finding the minimum cost Steiner tree, and think about what we did in class for the travelling salesman problem. Show that

$$\frac{|cost(SMT)|}{|cost(MST)|} \geq \frac{1}{2}$$

where SMT is the Steiner minimum tree on V', and MST is the minimum spanning tree on V'.

Question 3

Consider the following problem: you are given a set of obscure documents that you wish to classify into different topics. Unfortunately, you aren't really sure which topics each document falls into, seeing as the material is beyond your area of expertise. What you *do* have at your disposal though, is a function that, given two documents x and y, returns a '+' if x and y are similar, and a '$-$' if x and y are different, topic-wise. You would like to group, or *cluster* the documents into as many stacks as necessary so that related materials are in the same stacks, and any two unrelated documents are in different stacks. Unfortunately, this isn't always possible to do, so you decide you would like to cluster the documents in such a way that you make as few mistakes as possible, such as placing unrelated documents in the same stack, or two related documents in different stacks.

In particular, you are given a complete graph $G = (V, E)$, where each edge (u, v) is either labelled $+$ or $-$, depending on whether u and v are considered to be similar or different. Your goal is to produce a partition (clustering) of the vertices that agrees as much as possible with the edge labels. That is, we want a clustering that maximizes the number of agreements: the number of $+$ edges within clusters, plus the number of $-$ edges between clusters, or equivalently, minimizes the number of disagreements: the number of $-$ edges inside clusters plus the number of $+$ edges between clusters.

(a) A *perfect* clustering is a clustering that correctly places all the edges, i.e. all the positive edges are within clusters, and all negative edges cross between different clusters. Give a simple example demonstrating that a perfect clustering doesn't always exist.

(b) Suppose some little (honest) birdie flies by and tells you that there *is* perfect clustering of your graph G. Come up with an algorithm that outputs that perfect clustering.

(c) Part (a) tells us it is not always possible to obtain a perfect clustering, which is why we are concerned with finding clusters that maximize agreements or minimize disagreements. Give a 2-approximation for maximizing the number of agreements. I.e. an algorithm that produces a clustering that agrees with at least half the edge labels.

(d) Explain why your algorithm for (c) will not yield a 2-approximation for the objective of minimizing the number of disagreements, i.e. the number of $+$ edges between clusters plus the number of $-$ edges inside clusters.