

CS 482 Summer 2005
Homework Assignment #7

Out: August 2

Due: August 8

Question 1

Suppose we are given a graph $G = (V, E)$ on n nodes, and an integer k . Recall that the VERTEX COVER problem is as follows: Does there exist a subset of vertices $S \subseteq V$ such that $|S| = k$ and for every edge $e \in E$, one of its endpoints is in S . We will see in class that this problem is NP-Complete.

Now consider the following related problem. We are given a graph $G = (V, E)$, and would like to determine whether or not there is a vertex cover containing at most half of the vertices in V . We call this the MINORITY VERTEX COVER problem. Prove that MINORITY VERTEX COVER is NP-Complete.

Question 4

The teachers at Ithaca High recently got together and decided that their finals period is too damn long, and that from now on, all finals will be given on a single day. Naturally, the students rioted, and after a few bloody days a deal was worked out: Each student would be allowed to list a pair of tests that are not to be given on the same day. So Joe could request that History and Spanish not be on the same day, Mary could request that Math and Spanish not be on the same day, and Fred could request that English and Chemistry not be on the same day. Unfortunately, now they have all these restrictions, and they can't quite figure out how long a finals period they are going to need.

The TEST SCHEDULE problem is as follows: Given the list of finals f_1, f_2, \dots, f_n , a set P containing pairs of finals which are not to be scheduled at the same time, and an integer k , determine whether or not the n finals can be scheduled in k days without violating the restrictions listed in P . Prove that TEST SCHEDULE is NP-Complete.

Question 2

You are probably familiar with the board game *Clue*. The game begins with the shocking murder of Mr. Boddy, in his own mansion. As a player, you become a roving detective, sniffing out the answers to the classic mystery questions: Who among Mr. Boddy's guests is the murderer? (Miss Scarlett? Colonel Mustard? Professor Plum?) What was the murder weapon? (Lead pipe? Wrench? Or my weapon of choice, a candlestick?) Where did the murder happen? (Library? Conservatory?) Players travel from room to room on the board, making their guesses and ruling out the innocent. If you are the first to narrow it down to one suspect, one weapon and one room...you win.

Each suspect, weapon and room are pictured on playing cards. One from each stack is secretly placed in the "case file" at the beginning of the game, as the culprits, and the rest of the cards are mixed together and passed out amongst the players. Beginning with your own cards, you can start eliminating suspects. You make guesses to other players, and they will show you one of their cards if your suggestion is wrong.

You would like to use the information of your suggestions and the ones made by other players to rule out suspects: when player A makes a suggestion to player B , and A is shown a card, you know that one of the three cards guessed is *not* in the case file, but you don't know which one. One strategy for *Clue* could be to keep track of all incorrect suggestions in order to guess the minimum set of places, people, and weapons that appear in each of these guesses; this is the smallest set that could have been revealed, given the incorrect guesses. If you have such a minimum set and it leaves a single person, weapon, and place unselected, then you have a reasonable guess for an accusation.

More generally, suppose there are cards $C = \{c_1, c_2, \dots, c_n\}$ of places, weapons, and suspects. There are sets $B_1, B_2, \dots, B_m \subseteq C$, each B_i containing at least one revealed card. For example, for each card c either in your hand or shown to you by another player, there would be a singleton set $\{c\}$, and there would be $G = \{p, w, s\}$ for each incorrect guess G made by another player. Your goal is to determine if there is set $C' \subseteq C$ of size at most k such that $C \cap B_i \neq \emptyset$ for each B_i . Prove that this problem is NP-Complete.

For example, suppose you are dealt the wrench, the library, and Miss Scarlett, and there have been two incorrect guesses: Mr. Plum in the conservatory with the candlestick, and Mr. Plum in the kitchen with the rope. The minimum set of revealed cards is: the wrench, the library, Miss Scarlett, and Mr. Plum.

Disclaimer: This strategy doesn't necessarily work...it's just an attempt.

Question 3

For each of the following three problems, either prove it is NP-Complete, or give a polynomial time algorithm to solve it. If you want to show something is NP-Complete, feel free to use results from the previous questions on this assignment in your reduction.

Since November 23, 2004, you've been spending a significant amount of your time playing a little game called World of Warcraft, a multi-player fantasy adventure game.

(a) After days of fighting trolls and dragons, your backpack is full of loot you've acquired, which can only mean one thing: it's time to start crafting. In your backpack you have a set of ingredients $I = \{i_1, i_2, \dots, i_k\}$, where each $i_j \in I$ appears with multiplicity $m_j \geq 1$. You also have a list of items you want to make from these ingredients, where item r_ℓ requires ingredients $I_\ell \subseteq I$. Your goal is to determine how many items you can make, given your current set of ingredients.

The problem is as follows: Given the set of ingredients, their multiplicities, the items and their required ingredients, is there a way to select at least k items to make using only the ingredients available to you?

For example, suppose you have 2 units of silk, and one unit each of leather, copper, bronze, wool, ruby, sapphire, and dye. Then given the table below, it is possible for you to make 4 items: light armour, medium armour, fancy dagger, and heavy armour.

| Items | Required Ingredients | Items | Required Ingredients |
|--------------|-------------------------|------------------|----------------------|
| silk bag | silk, leather, sapphire | magical bandages | silk, sapphire |
| light armour | silk, wool, dye | medium armour | silk, leather |
| heavy armour | bronze, wool, sapphire | fancy dagger | copper, ruby |

(b) Having depleted your stores with all that crafting, it's time to replenish. You have a new list of items you want to make, using ingredients $I = \{i_1, i_2, \dots, i_k\}$, where you need $m_j \geq 1$ units of ingredient i_j . You have none of the ingredients right now, but they can be obtained through the completion of certain quests, where quest q_ℓ provides ingredients I_ℓ . Your goal is to determine how many quests you need to complete in order to obtain all of your desired ingredients.

The problem is as follows: Given the set of ingredients, their required multiplicities, the quests and their provided ingredients, is there a way to select at most k quests such that each you obtain at least m_j units of ingredient i_j ?

For example, you want to get 1 unit of each ingredient other than silk, of which you want 2 units. Then it is possible to get these ingredients by completing the 4 quests Slay the Dragon, Protect the Traveling Merchant, Explore the Haunted Island, and Bring Lancelot Beer.

| Quest | Provided Ingredients | Armour/Reward |
|--------------------------------|----------------------|---------------------|
| Slay the Dragon | silk, leather | gloves, boots |
| Kill 12 Pirates | ruby, dye | gloves, shield |
| Find the Lost Wizard | bronze | gloves, chest piece |
| Bring Lancelot Beer | wool, ruby, bronze | chest piece, helm |
| Explore the Haunted Island | dye, sapphire | shield, boots |
| Protect the Traveling Merchant | silk, dye | shield, boots |

(c) Ten levels later, the armour you've made is started to look a little worse-for-wear. In order to upgrade your armour, you need to complete quests where the desired armour is offered as a reward for the quest's completion. You have a list of desired armour $A = \{a_1, a_2, \dots, a_n\}$, and a set $Q = \{q_1, q_2, \dots, q_m\}$ of potential quests. Quest q_i offers a set A_i of armour you could receive as a reward, but the catch is that for each quest, you can *only select one* of the possible armour as your reward. Your goal is to determine which quests to go on to collect all the desired armour.

The problem is as follows: Given the set of desired armour, quests, and each quest's potential rewards, is there a way to select at most k quests such that you obtain all your desired armour?

For example, you would need to go on the quest Bring Lancelot Beer to obtain the helm, then Find the Lost Wizard to obtain the chest piece. You could then Slay the Dragon, Kill 12 Pirates, and Explore the Haunted Island to obtain the remaining gloves, shield, and boots.