CS 482 Summer 2005

**Homework Assignment #5**

Out: July 26 Due: July 29

## Question 1

Let $f_1$ and $f_2$ be flows in a network $G = (V, E)$, and let $\alpha$ and $\beta$ be real numbers. The *scalar flow product* $\alpha f_1$ is a function from $E$ to $\Re$ defined by

$$(\alpha f_1)(e) = \alpha \cdot f_1(e),$$

whereas the *additive flow* $\alpha f_1 + \beta f_2$ is a function from $E$ to $\Re$ defined by

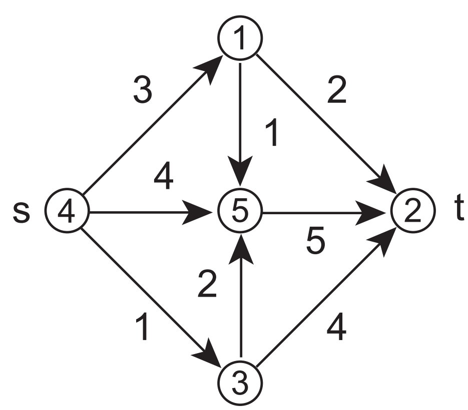
$$(\alpha f_1 + \beta f_2)(e) = \alpha \cdot f_1(e) + \beta \cdot f_2(e).$$

(a) For which values of $\alpha$ is $\alpha f_1$ still a flow?

(b) For which values of $\alpha$ and $\beta$ is $\alpha f_1 + \beta f_2$ still a flow?

## Question 2

Consider the following network.




(a) List all the $s$-$t$ cuts, and calculate their capacities.

(b) Run the Ford-Fulkerson algorithm to find the maximum flow, starting with the flow $f_{43} = 1, f_{35} = 1, f_{52} = 1$, and all other $f_{ij} = 0$. Show your intermediate residual graphs.

## Question 3

Consider a flow network $G = (V, E)$ with capacities $c(e)$ on each edge $e$. Let $r(e)$ be the amount by which you can decrease the capacity of edge $e$ and not affect the value of $G$'s maximum flow. (In particular, $r(e) = 0$ for all min-cut edges $e$ and $r(f) > 0$ for all non-min-cut edges $f$.) Let $e_1, e_2, \ldots, e_k \in E$ be $k$ edges with $r(e_i) > 0$, $1 \le i \le k$. Let $r_{\min}(e_1, \ldots, e_k) = \min_{1 \le i \le k}\{r(e_i)\}$. Show that we can remove $r_{\min}/k$ capacity from each $e_i$, $1 \le i \le k$ (i.e. set $c(e_i) = c(e_i) - r_{\min}/k$) and not affect the value of $G$'s maximum flow.

## Question 4

Let $G = (V, E)$ be our usual network with designated nodes $s$ and $t$, but without the restrictions on $s$ and $t$. In particular, $G$ is a directed graph with capacity $c_e > 0$ for edge $e$ and vertices $s$ and $t$, but $s$ may have in-edges and $t$ may have out-edges.

Define $E_{out}(v)$ to be the out-edges of a vertex $v$, i.e. $E_{out}(v) = \{(v, w) | (v, w) \in E\}$, and define $E_{in}(v)$ to be the in-edges of a vertex $v$, i.e. $E_{in}(v) = \{(w, v) | (w, v) \in E\}$.

Prove the following two statements:

(1) There exists a maximum $s$-$t$ flow $f$ such that $\forall e \in E_{out}(t)$, $f_e = 0$ (i.e. no flow leaving sink).

(2) There exists a maximum $s$-$t$ flow $f$ such that $\forall e \in E_{in}(s)$, $f_e = 0$ (i.e. no flow into source).

**Hint:** Try using the Max-Flow Min-Cut theorem.