

CS 482 Summer 2004  
**Proof Techniques: Induction**

## Main Steps

The 5 main steps for an induction proof are as follows:

**Step 1: State your  $P(n)$ .** State your  $P(n)$ , which should be a property as a function of  $n$ . Also state for which  $n$  you will prove your  $P(n)$  to be true.

**Step 2: State your base case.** State for which  $n$  your base case is true, and prove it.

**Step 3: State your induction hypothesis.** State your induction hypothesis. State your induction hypothesis! Without it, the whole proof falls apart. Usually it is just restating your  $P(n)$  but with a quantifier in front (such as, for all  $k \geq 0$ , assume  $P(k)$  is true).

**Step 4: Inductive Step.** This is where you try to prove a larger case of the problem than you assumed in your induction hypothesis. What are you trying to prove? Keep this in mind when you do this step. Remember, use your induction hypothesis somewhere, and tell us where. If you haven't used your induction hypothesis in the step, then you are not doing a proof by induction. So you'd better need to use it.

**Step 5: Conclusion.** This is optional. You can re-state the problem.

## Comments

- If your  $P(n)$  doesn't mention  $n$  in it anywhere, there's trouble.
- $P(n)$  is a *property*, not a number, so you *cannot* manipulate it mathematically, like  $P(n) = 5$ , or  $P(n + 1) < P(n)$ .
- Be careful with the base case... sometimes you will need more than one, as with some recurrence relations.

- Recall the difference between strong and weak induction. They're equivalent, but sometimes using one is easier than using the other.
- Induction is often used to prove properties of natural numbers or recursive structures, or properties of structures that are easily decomposed into substructures sharing the same property.

## Example

For  $n \geq 0$ , let  $P(n)$  be the statement that  $\sum_{i=1}^n 2^i = 2^{n+1} - 1$ .

**Base Case:** Let  $n = 0$ . Then we want to show that  $\sum_{i=0}^0 2^i = 2^1 - 1$ . But

$$\begin{aligned}
 LHS &= \sum_{i=0}^0 2^i \\
 &= 2^0 \\
 &= 1 \\
 &= 2 - 1 \\
 &= 2^1 - 1 = RHS
 \end{aligned}$$

**Induction Hypothesis:** Assume  $P(k)$  is true for some  $k \geq 0$ . I.e. assume that  $\sum_{i=1}^k 2^i = 2^{k+1} - 1$ .

**Inductive Step:** We want to prove  $P(k+1)$ , i.e. that  $\sum_{i=1}^{k+1} 2^i = 2^{k+2} - 1$ .

$$\begin{aligned}
 LHS &= \sum_{i=1}^{k+1} 2^i \\
 &= 2^{k+1} + \sum_{i=1}^k 2^i \\
 &\stackrel{IH}{=} 2^{k+1} + (2^{k+1} - 1) \\
 &= 2 \cdot 2^{k+1} - 1 \\
 &= 2^{k+2} - 1 = RHS
 \end{aligned}$$

**Conclusion:** By mathematical induction, we have shown that for all  $n \geq 0$ ,  $\sum_{i=1}^n 2^i = 2^{n+1} - 1$ .