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Decision problems ($\{0,1\}$-valued functions)

- all instances
- "yes" instances

Possible inputs: strings over a finite alphabet

- Alphabet - any finite set
  Ex: $\{0,1\}$ $\{a,b\}$ $\{0,1,2,\ldots,9\}$
- Σ elements are <u>symbols</u> or <u>letters</u> $a, b, c, \ldots$

<u>Strings</u> are finite sequences ~~from~~ Σ.

Ex: $\Sigma = \{a,b\}$ aabab a string of length 5

$|aabab| = 5$

- ε = <u>null string</u> or <u>empty string</u> $|\varepsilon| = 0$
- a^n = string of a's of length n $(n \in \mathbb{N})$

$|aaaaa| = 5 = |a^5|$

$(\mathbb{N} = \{0,1,2,\ldots\})$

$a^0 \triangleq \varepsilon \quad a^{n+1} \triangleq a^n a \quad \stackrel{\text{def}}{=}$

- Set of all strings over Σ is denoted Σ^*

Ex: $\{a,b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$

$\{a\}^* = \{\varepsilon, a, aa, \ldots\} = \{a^n \mid n \geq 0\}$

$\varnothing^* = \{\varepsilon\}$ by convention

$|\varepsilon| = 0$ $\quad$ $|xa| = |x| + 1$

$|\cdot| : \Sigma^* \to \mathbb{N}$

$\{a, b\} = \{b, a\}$ $\quad$ $ab \neq ba$

$\{a, a, b\} = \{a, b\}$ $\quad$ $aab \neq ab$

$\cancel{\{a, a, b\} \neq \{a, b\} = \{b, a\}}$

Concatenation $\cdot : \Sigma^* \times \Sigma^* \to \Sigma^*$

$xy \in \Sigma^*$ if $x \in \Sigma^*$ & $y \in \Sigma^*$

- $(xy)z = x(yz)$
- $\varepsilon x = x \varepsilon = x$
- $|xy| = |x| + |y|$
- $|a^n| = n$
- $a^m a^n = a^{m+n}$
- $|\varepsilon| = 0$

$(M, \cdot, 1)$ where $\cdot$ is associative
1 is a 2-sided identity
is called a <u>monoid</u> (semigroup w/ ident)

$(\Sigma^*, \cdot, \varepsilon)$ is a monoid

$(\mathbb{N}, +, 0)$ " " "

$|\cdot| : \Sigma^* \to \mathbb{N}$ is a monoid homomorphism

- operations on sets $A, B \subseteq \Sigma^*$

$A \cup B \quad A \cap B \quad \sim A = \{x \in \Sigma^* \mid x \notin A\}$

$= \Sigma^* - A = \Sigma^* \setminus A$

- $A \subseteq B \overset{\triangle}{\Leftrightarrow} \forall x \; x \in A \Rightarrow x \in B$
- $A = B \overset{\triangle}{\Leftrightarrow} \forall x \; x \in A \Leftrightarrow x \in B$

- $AB \triangleq \{xy \mid x \in A \;\&\; y \in B\}$

  set concatenation $\cdot : 2^{\Sigma^*} \times 2^{\Sigma^*} \to 2^{\Sigma^*}$

  $\wp(\Sigma^*) \times \wp(\Sigma^*) \to \wp(\Sigma^*)$

Ex: $\{a, ab\} \cdot \{b, ba\} = \{ab, aba, abb, abba\}$

- $\{\varepsilon\}$ is an identity $\{\varepsilon\} \cdot A = A \cdot \{\varepsilon\} = A$
- $(AB)C = A(BC)$

  $(2^{\Sigma^*}, \cdot, \{\varepsilon\})$ forms a monoid
- powers: $A^n = \underbrace{A \cdot A \cdot \ldots A}_{n}$

$A^0 \triangleq \{\varepsilon\} \quad A^{n+1} \triangleq A^n \cdot A$

- $A^* \triangleq \bigcup_{n \geq 0} A^n = A^0 \cup A^1 \cup A^2 \cup \ldots$

$= \{x_1 x_2 \cdots x_n \mid n \geq 0 \;\&\; x_i \in A, 1 \leq i \leq n\}$

Kleene asterate

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \qquad A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

$$\{a, ab\} \cup \{ab, aab\} = \{a, ab, aab\} \qquad \{a, ab\} \cap \{ab, aab\} = \{ab\}$$

$$\sim A = \{x \in \Sigma^* \mid x \notin A\} = \Sigma^* - A = \Sigma^* \setminus A$$

$$\begin{aligned} AB &= \{xy \mid x \in A \text{ and } y \in B\} \\ \{a, ab\}\{b, ba\} &= \{ab, aba, abb, abba\} \end{aligned}$$

$$A^0 = \{\varepsilon\} \qquad A^{n+1} = AA^n$$

$$\begin{aligned} \{ab, aab\}^0 &= \{\varepsilon\} \\ \{ab, aab\}^1 &= \{ab, aab\} \\ \{ab, aab\}^2 &= \{abab, abaab, aabab, aabaab\} \\ \{ab, aab\}^3 &= \{ababab, ababaab, abaabab, aababab, \\ &\quad\; abaabaab, aababaab, aabaabab, aabaabaab\} \end{aligned}$$

$$\{a, b\}^n = \{x \in \{a, b\}^* \mid |x| = n\}$$

$$\begin{aligned} A^* &= \bigcup_{n \geq 0} A^n = A^0 \cup A^1 \cup A^2 \cup A^3 \cup \cdots \\ &= \{x_1 x_2 \cdots x_n \mid n \geq 0 \text{ and } x_i \in A,\ 1 \leq i \leq n\} \end{aligned}$$

$$A^+ = AA^* = \bigcup_{n \geq 1} A^n$$

Associativity:

$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A \cap B) \cap C = A \cap (B \cap C) \qquad (AB)C = A(BC)$$

Commutativity:

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Identities:

$$A \cup \varnothing = \varnothing \cup A = A \qquad \{\varepsilon\}A = A\{\varepsilon\} = A$$

Annihilation:

$$A\varnothing = \varnothing A = \varnothing$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A(B \cup C) = AB \cup AC \qquad (A \cup B)C = AC \cup BC$$

$$A(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} AB_i \qquad (\bigcup_{i \in I} B_i)A = \bigcup_{i \in I} B_i A$$

De Morgan laws:

$$\sim(A \cup B) = \sim A \cap \sim B \qquad \sim(A \cap B) = \sim A \cup \sim B$$

Some properties of *:

$$A^* A^* = A^* \qquad (A^*)^* = A^* \qquad A^* = \{\varepsilon\} \cup AA^* = \{\varepsilon\} \cup A^* A \qquad \varnothing^* = \{\varepsilon\}$$

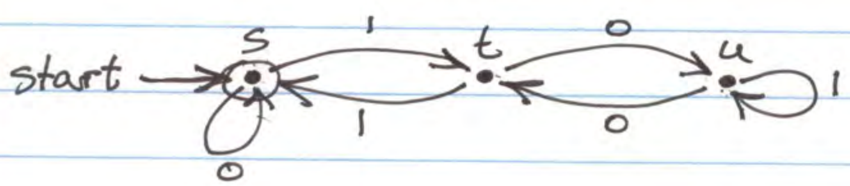
Deterministic finite automata (DFAs)
(singular: automaton, plural: automata)

A DFA is a 5-tuple $M = (Q, \Sigma, \delta, s, F)$ where:

- Q is a finite set (the <u>states</u>)
- Σ is a finite set (the <u>input alphabet</u>)
- s is the <u>start state</u>, $s \in Q$
- F is the set of <u>accept states</u> or <u>final states</u>, $F \subseteq Q$
- $\delta : Q \times \Sigma \longrightarrow Q$ is the <u>transition function</u>
  $\delta(q, a)$ is the state to move to next when in state q, seeing input symbol a.

To specify,
- write down components
- draw a diagram

$110101 \in \Sigma^*$
$1001 \in \Sigma^*$



$\Sigma = \{0,1\}$ start state is s, denoted by $\rightarrow$
$F = \{s\}$, denoted by a circle

- table:

	0	1
$\rightarrow s F$	s	t
t	u	s
u	t	u