Machine Learning for Intelligent Systems

Lecture 7: Convergence of Perceptron

Reading: UML 9.1

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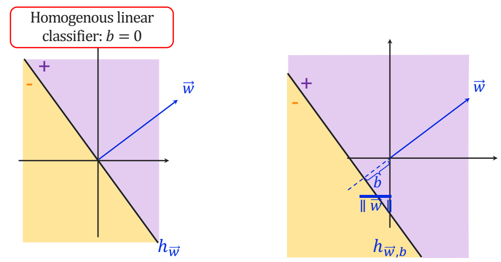
Linear Classifiers

For a vector $\vec{w} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, the hypothesis $h_{\vec{w},b}: \mathbb{R}^d \to \mathbb{R}$ defined bellow is called a **linear classifier/linear predictor/halfspace**

$$h_{\vec{w},b}(\vec{x}) = sign(\vec{w} \cdot \vec{x} + b) = \begin{cases} +1 & \vec{w} \cdot \vec{x} + b > 0 \\ -1 & \vec{w} \cdot \vec{x} + b \leq 0 \end{cases}$$



Homogenous vs. Non-homogenous

Any d-dimensional learning problem for **non-homogenous linear classifiers** has a **homogenous** form in (d+1) dimension.

Non-Homogenous $HS^d = \{h_{\vec{w},b} \mid \vec{w} \in \mathbb{R}^d, b \in \mathbb{R}\}$	Homogenous $HS^{d+1}_{homogenous} = \{h_{\vec{w}'} \mid \vec{w}' \in \mathbb{R}^{d+1}\}$
$\vec{x}$	$\vec{x}' = (\vec{x}, +1)$
$\vec{w}, b$	$\vec{w}' = (\vec{w}, b)$
$\vec{w} \cdot \vec{x} + b$	$\vec{w}' \cdot \vec{x}' = \vec{w} \cdot \vec{x} + b$

Without loss of generality, focus on **homogenous linear classifiers.**

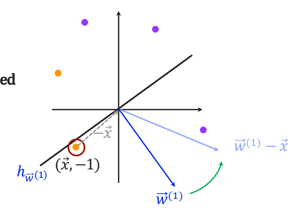
Improving a linear classifier

If there is a homogeneous linear classifier that is consistent with $\{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$, how can we find it?

Last time: Do it with a linear program

This time: Start with a guess and improve it.

Move away from misclassified negative points: $\vec{w} - \vec{x}$



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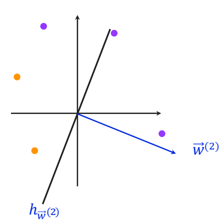

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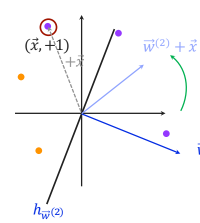

Move towards misclassified positive points: $\vec{w} + \vec{x}$

Improving a linear classifier

If there is a homogeneous linear classifier that is consistent with $\{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$, how can we find it?

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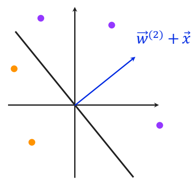
Move away from misclassified negative points: $\vec{w} - \vec{x}$



Move towards misclassified positive points: $\vec{w} + \vec{x}$

Perceptron (homogeneous & batch)

Input: Training data set $\{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$

Initialize $\vec{w}^{(0)} = (0, \ldots, 0)$, $t = 0$

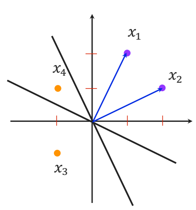
While there is $i \in [m]$, such that $\underbrace{y_i\left(\vec{w}^{(t)} \cdot \vec{x}_i\right) \leq 0}_{\text{misclassified}}$ then,

- $\vec{w}^{(t+1)} = \vec{w}^{(t)} + y_i\, \vec{x}_i$ $\begin{cases} \vec{w}^{(t)} + \vec{x}_i & \text{for positive instances} \\ \vec{w}^{(t)} - \vec{x}_i & \text{for negative instances} \end{cases}$
- $t \leftarrow t + 1$

End While

Output $\vec{w}^{(t)}$

Example



	x_1	x_2	y
$\vec{x}_1 =$	(1	2)	$y_1 = 1$
$\vec{x}_2 =$	(2	1)	$y_2 = 1$
$\vec{x}_3 =$	(-1	-1)	$y_3 = -1$
$\vec{x}_4 =$	(-1	1)	$y_3 = -1$

- $\vec{w}^{(0)} = (0, 0)$
  - → $y_1(\vec{w}^{(0)} \cdot \vec{x}_1) = 0 \leq 0$
- $\vec{w}^{(1)} = \vec{w}^{(0)} + x_1 = (1,2)$
  - → $y_1(\vec{w}^{(1)} \cdot \vec{x}_1) = 5 > 0$
  - → $y_2(\vec{w}^{(1)} \cdot \vec{x}_2) = 4 > 0$
  - → $y_3(\vec{w}^{(1)} \cdot \vec{x}_3) = 3 > 0$
  - → $y_4(\vec{w}^{(1)} \cdot \vec{x}_4) = -1 \leq 0$
- $\vec{w}^{(2)} = \vec{w}^{(1)} - x_4 = (2,1)$

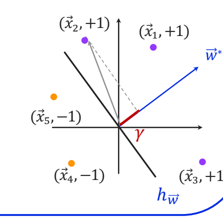
Margin & Convergence

Margin

Given a data set $S = \{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$ and a linear classifier $h_{\vec{w}}$ that is consistent with S, that is, $y_i(\vec{w} \cdot \vec{x}_i) > 0$, the *geometric margin* of $h_{\vec{w}}$ is defined as:

$$\gamma \coloneqq \min_{i \in S} \frac{y_i(\vec{w} \cdot \vec{x}_i)}{\| \vec{w} \|}$$

Margin γ is the distance of the closest instance to hyperplane $\vec{w} \cdot \vec{x} = 0$.



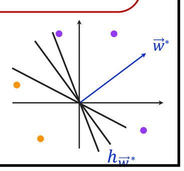
Convergence of Perceptron

Theorem: Convergence of Perceptron

Given a data set $S = \{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$ and radius R such that $\| \vec{x}_i \| \leq R$ for all $i \in [m]$.

If $h_{\vec{w}^*}$ is consistent with S with margin $\gamma \coloneqq \min_{i \in S} \frac{y_i(\vec{w}^* \cdot \vec{x}_i)}{\|\vec{w}^*\|}$ then Perceptron makes at most R^2/γ^2 updates before predicting every label perfectly.

Idea: $h_{\vec{w}^*}$ has γ margin → there is wiggle room.
→ **Show that within** $t = R^2/\gamma^2$,
$\vec{w}^t$ is close to $\vec{w}^*$ in angle.



Proof Ideas

Theorem: Convergence of Perceptron

Perceptron makes at most R^2/γ^2 updates before predicting every label perfectly. Recall margin $\gamma \coloneqq \min_{i \in S} \frac{y_i(\vec{w}^* \cdot \vec{x}_i)}{\|\vec{w}^*\|}$.

Idea: Proof by Contradiction. Show that if within $t > R^2/\gamma^2$,

$$\cos(\,\theta(\vec{w}^*, \vec{w}^{(t+1)})) = \frac{\vec{w}^* \cdot \vec{w}^{(t+1)}}{\| \vec{w}^* \| \| \vec{w}^{(t+1)} \|} > 1 \qquad \textbf{Impossible}$$

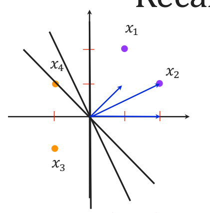
Plan:

- Assume $\vec{w}^*$ is normalized to be unit vector → margin doesn't change.

1. Show that $\vec{w}^* \cdot \vec{w}^{(t+1)}$ is large (find a lower bound).
2. Show that $\| \vec{w}^{(t+1)} \|$ is not too large (find an upper bound).

→ So, if $t > R^2/\gamma^2$, the cosine will be larger than 1 → Contradiction.

Recall: Example



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- $\vec{w}^{(0)} = (0, 0)$
  $\rightarrow y_1(\vec{w}^{(0)} \cdot \vec{x}_1) = 0$
- $\vec{w}^{(1)} = \vec{w}^{(0)} + x_1 = (1,2)$
  $\rightarrow y_1(\vec{w}^{(1)} \cdot \vec{x}_1) = 5 > 0$
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- $\vec{w}^{(2)} = \vec{w}^{(1)} - x_4 = (2,1)$

- Update on (x_2, y_2)
  $\rightarrow \vec{w}^{(1)} = (2, 1)$ converges in 1 step.
- Update on (x_3, y_3)
  $\rightarrow \vec{w}^{(1)} = (1, 1)$
  Update on (x_4, y_4)
  $\rightarrow \vec{w}^{(2)} = (2, 0)$

Online Perceptron

Theorem: Mistake Bound of Online Perceptron

Given a sequence of data $(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)$ one by one, with radius R and margin $\gamma \coloneqq \min_{i \in S} \frac{y_i(\vec{w}^* \cdot \vec{x}_i)}{\|\vec{w}^*\|}$ for some $\vec{w}^*$.

Online prediction: At each time use the current $\vec{w}$ to predict the label of incoming $(\vec{x}_i, y_i)$, update if needed.

Mistake Bound: The number of mistake that perceptron makes is at most R^2/γ^2.

Example: Reuters Text Classification