Optimal Hyperplanes

CS478 – Machine Learning
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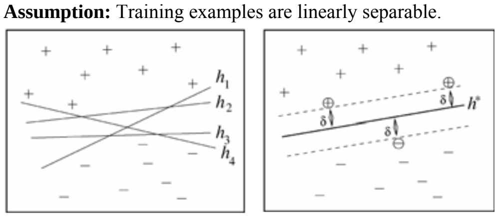
Reading: Schoelkopf/Smola Chapter 7.1-7.3, 7.5 (online)

Outline

- **Optimal hyperplanes and margins**
- **Hard-margin Support Vector Machine**
- **Primal optimization problem**
- **Soft-margin Support Vector Machine**

Optimal Hyperplanes
Linear Hard-Margin Support Vector Machine

Assumption: Training examples are linearly separable.

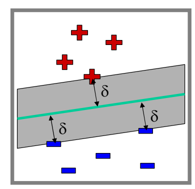


Hard-Margin Separation

Goal: Find hyperplane with the largest distance to the closest training examples.

Optimization Problem (Primal):

$$\min_{\vec{w},b} \quad \frac{1}{2}\vec{w}\cdot\vec{w}$$
$$s.t. \quad y_1(\vec{w}\cdot\vec{x}_1+b)\geq 1$$
$$\ldots$$
$$y_n(\vec{w}\cdot\vec{x}_n+b)\geq 1$$

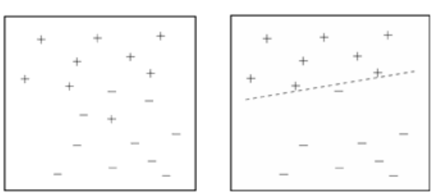


Support Vectors: Examples with minimal distance (i.e. margin).

Non-Separable Training Data

Limitations of hard-margin formulation

- For some training data, there is no separating hyperplane.
- Complete separation (i.e. zero training error) can lead to suboptimal prediction error.



Soft-Margin Separation

Idea: Maximize margin and minimize training error.

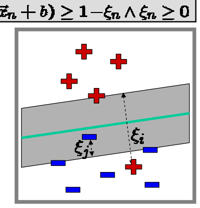
Hard-Margin OP (Primal):

$$\min_{\vec{w},b} \quad \frac{1}{2}\vec{w}\cdot\vec{w}$$
$$s.t. \quad y_1(\vec{w}\cdot\vec{x}_1+b)\geq 1$$
$$\ldots$$
$$y_n(\vec{w}\cdot\vec{x}_n+b)\geq 1$$

Soft-Margin OP (Primal):

$$\min_{\vec{w},\vec{\xi},b} \quad \frac{1}{2}\vec{w}\cdot\vec{w}+C\sum_{i=1}^{n}\xi_i$$
$$s.t. \quad y_1(\vec{w}\cdot\vec{x}_1+b)\geq 1-\xi_1 \wedge \xi_1\geq 0$$
$$\ldots$$
$$y_n(\vec{w}\cdot\vec{x}_n+b)\geq 1-\xi_n \wedge \xi_n\geq 0$$

- Slack variable ξ_i measures by how much (x_i,y_i) fails to achieve margin δ
- $\Sigma\xi_i$ is upper bound on number of training errors
- C is a parameter that controls trade-off between margin and training error.



Controlling Soft-Margin Separation

- $\Sigma\xi_i$ is upper bound on number of training errors
- C is a parameter that controls trade-off between margin and training error.

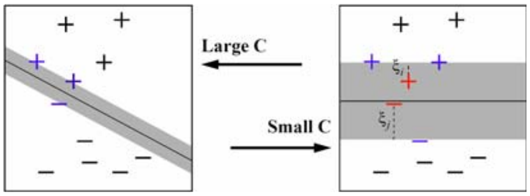
Soft-Margin OP (Primal):

$$\min_{\vec{w},\vec{\xi},b} \frac{1}{2}\vec{w}\cdot\vec{w} + C\sum_{i=1}^{n}\xi_i$$

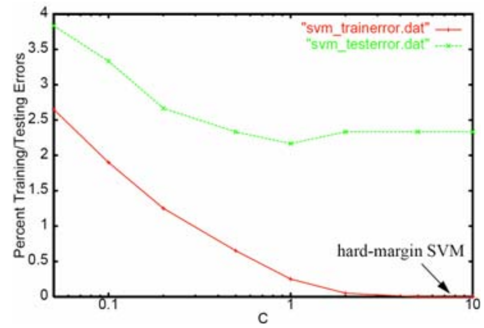
$$s.t.\ y_1(\vec{w}\cdot\vec{x}_1 + b) \geq 1-\xi_1 \wedge \xi_1 \geq 0$$

$$\ldots$$

$$y_n(\vec{w}\cdot\vec{x}_n + b) \geq 1-\xi_n \wedge \xi_n \geq 0$$



Example Reuters “acq”: Varying C



Example: Margin in High-Dimension

Training Sample S_{train}	$\vec{x}$ x_1	x_2	x_3	x_4	x_5	x_6	x_7	*y*
$(\vec{x}_1, y_1)$	1	0	0	1	0	0	0	1
$(\vec{x}_2, y_2)$	1	0	0	0	1	0	0	1
$(\vec{x}_3, y_3)$	0	1	0	0	0	1	0	-1
$(\vec{x}_4, y_4)$	0	1	0	0	0	0	1	-1
	$\vec{w}$ w_1	w_2	w_3	w_4	w_5	w_6	w_7	**b**
Hyperplane 1	1	1	0	0	0	0	0	2
Hyperplane 2	0	0	0	1	1	-1	-1	0
Hyperplane 3	1	-1	1	0	0	0	0	0
Hyperplane 4	0.5	-0.5	0	0	0	0	0	0
Hyperplane 5	1	-1	0	0	0	0	0	0
Hyperplane 6	0.95	-0.95	0	0.05	0.05	-0.05	-0.05	0