# Hidden Markov Models

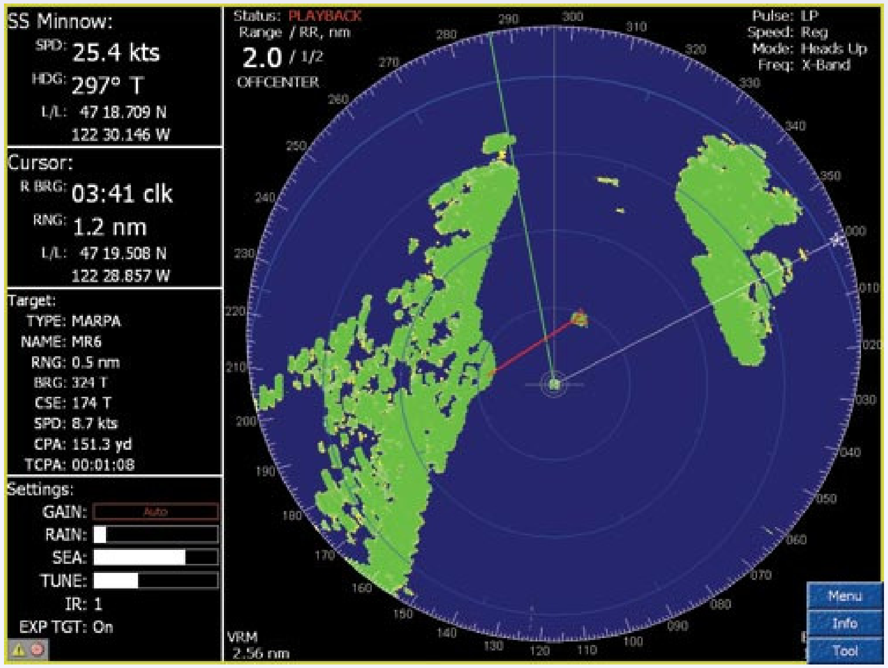
*CS4758: Robot Learning*

Ashutosh Saxena

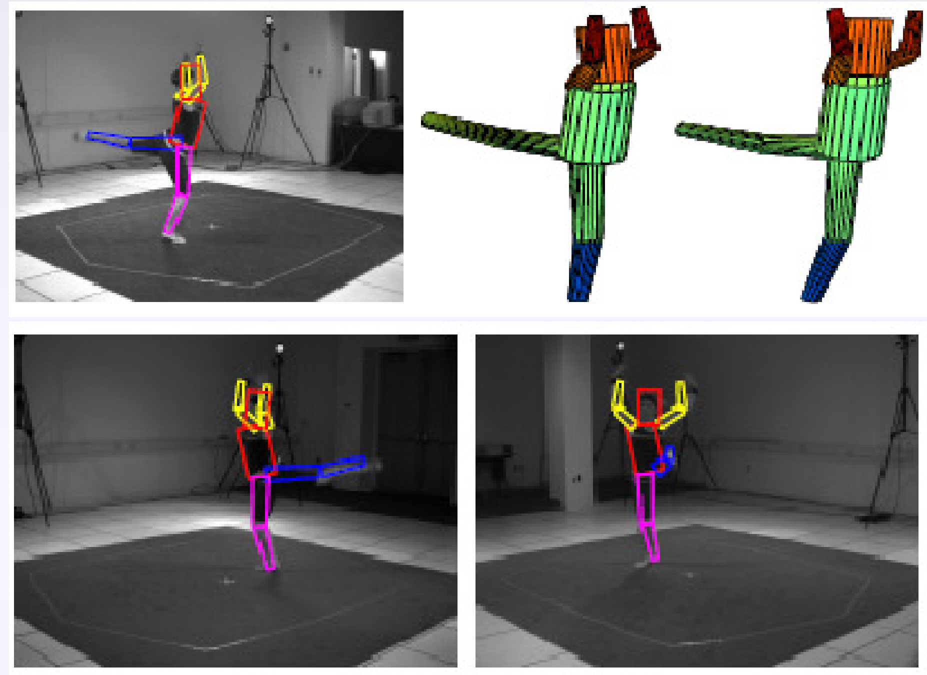
Lecture 14

Lecture slides also taken from Eric Sudderth and Andrew Moore.

# Target Tracking




*Radar-based tracking of multiple targets*



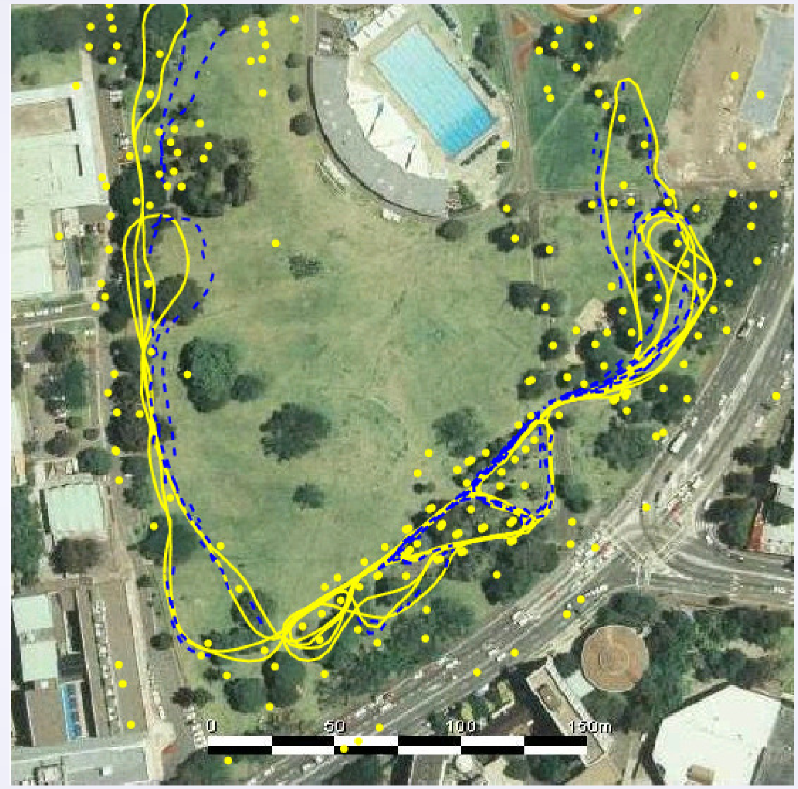
*Visual tracking of articulated objects*
*(L. Sigal et. al., 2006)*

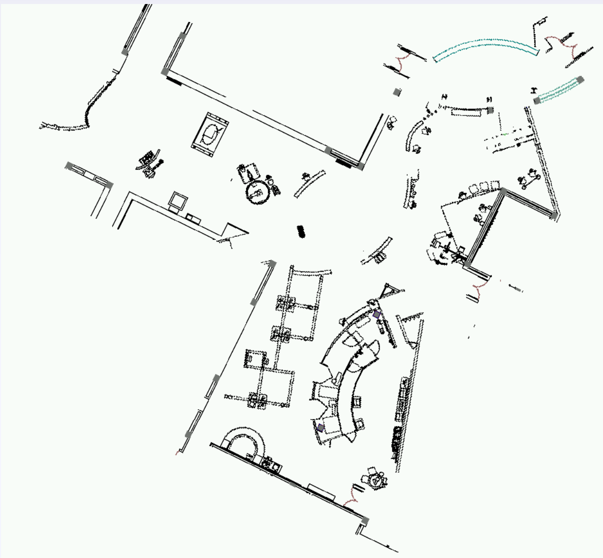
- Estimate motion of targets in 3D world from indirect, potentially noisy measurements

# Robot Navigation: *SLAM*

*Simultaneous Localization and Mapping*

*Landmark SLAM*
*(E. Nebot, Victoria Park)*



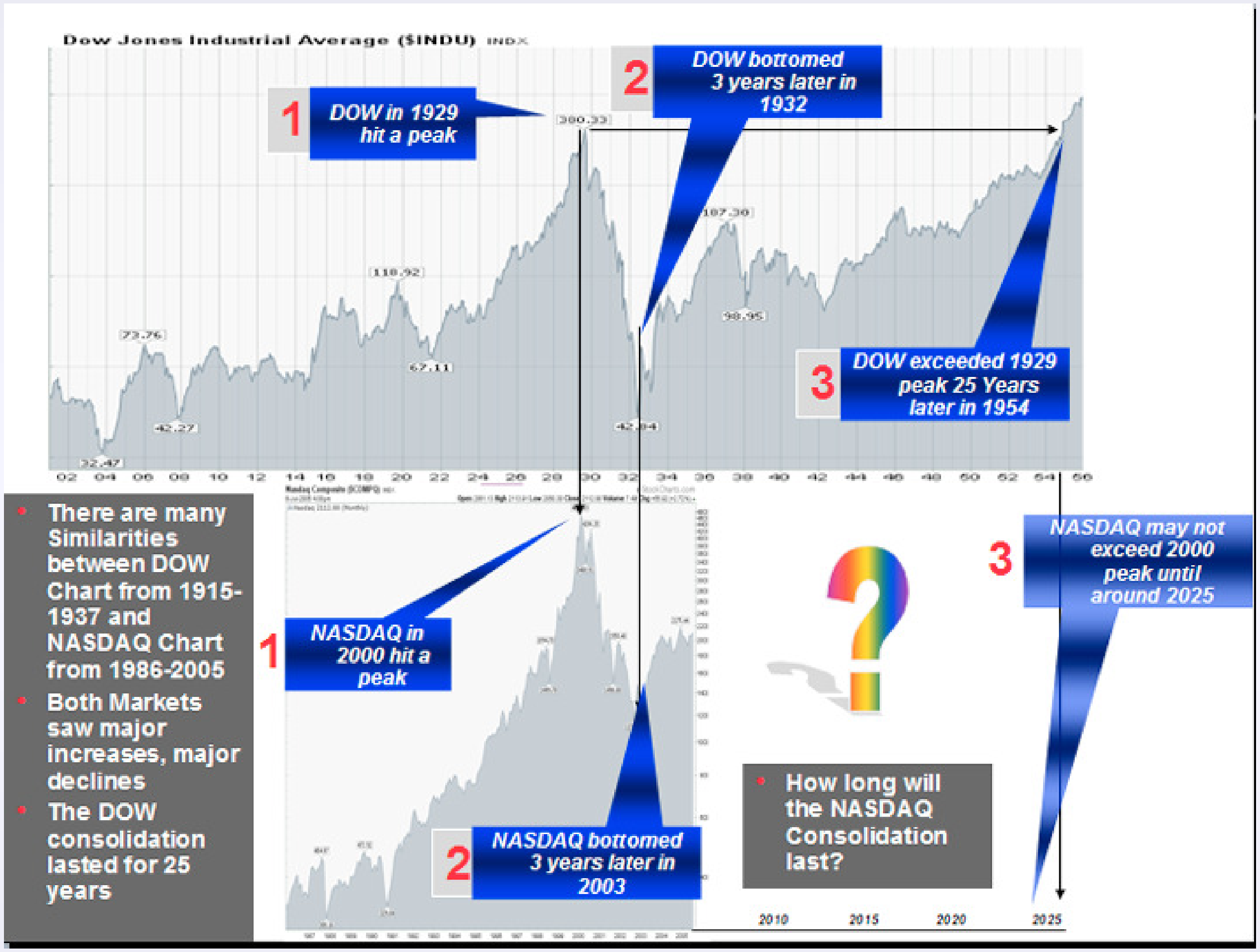
*CAD Map*

*(S. Thrun, San Jose Tech Museum)*

*Estimated Map*

- As robot moves, estimate its pose & world geometry

# Financial Forecasting




*http://www.steadfastinvestor.com/*

- Predict future market behavior from historical data, news reports, expert opinions, …

# Outline

**Introduction to Temporal Processes**

- Markov chains
- Hidden Markov models

**Discrete-State HMMs**

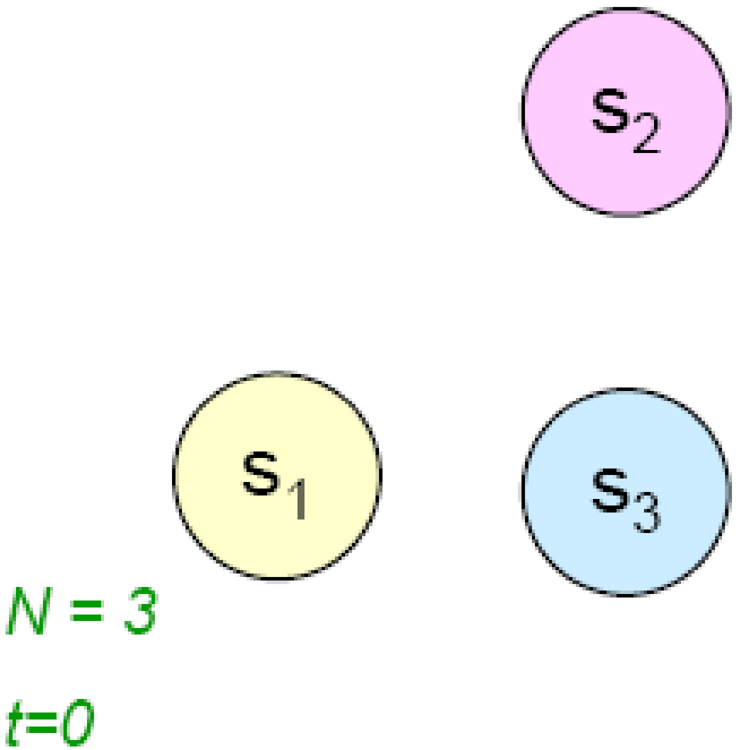
- Inference:  Filtering, smoothing, Viterbi, classification
- Learning:  EM algorithm

**Continuous-State HMMs**

- Linear state space models:  Kalman filters
- Nonlinear dynamical systems:  Particle filters

**Applications and Extensions**

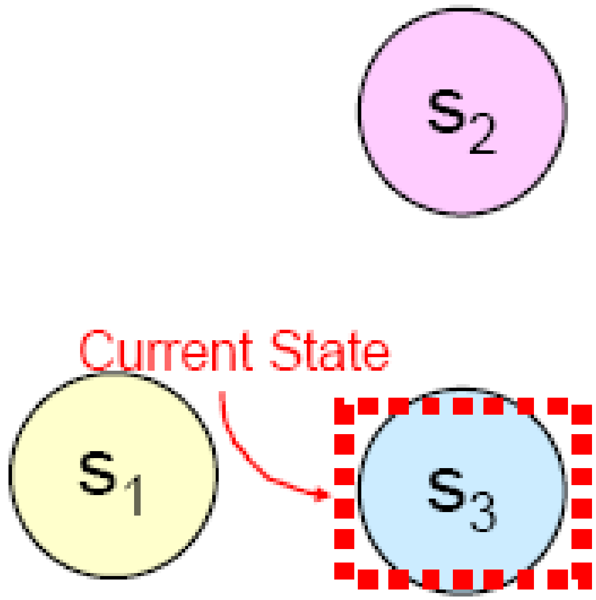
# A Markov System




Has $N$ states, called $s_1$, $s_2$ .. $s_N$

There are discrete timesteps, $t=0$, $t=1$, …

# A Markov System




$N = 3$

$t=0$

$q_t=q_0=s_3$

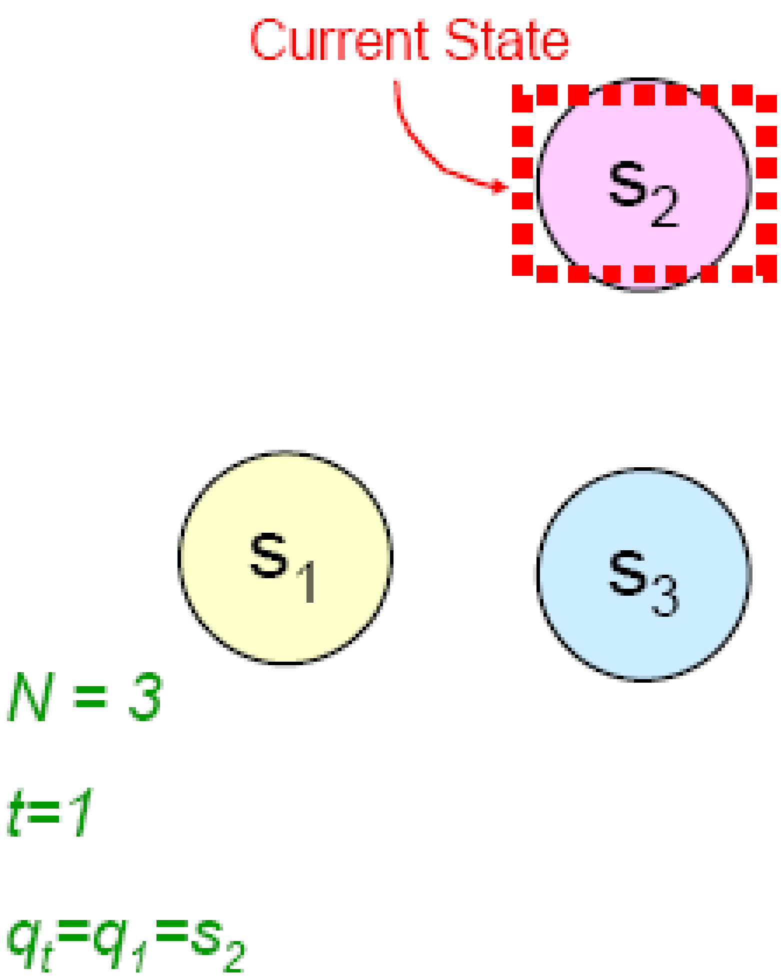
Has $N$ states, called $s_1$, $s_2$ .. $s_N$

There are discrete timesteps, $t=0$, $t=1$, …

On the t'th timestep the system is in exactly one of the available states. Call it $q_t$

Note: $q_t \in \{s_1, s_2 .. s_N\}$

# A Markov System




Has $N$ states, called $s_1, s_2 .. s_N$

There are discrete timesteps, $t=0, t=1, \dots$

On the t'th timestep the system is in exactly one of the available states. Call it $q_t$

Note: $q_t \in \{s_1, s_2 .. s_N \}$

Between each timestep, the next state is chosen randomly.

# A Markov System

$P(q_{t+1}=s_1|q_t=s_2) = 1/2$

$P(q_{t+1}=s_2|q_t=s_2) = 1/2$

$P(q_{t+1}=s_3|q_t=s_2) = 0$

$P(q_{t+1}=s_1|q_t=s_1) = 0$

$P(q_{t+1}=s_2|q_t=s_1) = 0$

$P(q_{t+1}=s_3|q_t=s_1) = 1$

$s_2$

$s_1$

$s_3$

$N = 3$

$t=1$

$q_t=q_1=s_2$

$P(q_{t+1}=s_1|q_t=s_3) = 1/3$

$P(q_{t+1}=s_2|q_t=s_3) = 2/3$

$P(q_{t+1}=s_3|q_t=s_3) = 0$

Has $N$ states, called $s_1, s_2 \,..\, s_N$

There are discrete timesteps, $t=0$, $t=1$, …

On the t'th timestep the system is in exactly one of the available states. Call it $q_t$

Note: $q_t \in \{s_1, s_2 \,..\, s_N\}$

Between each timestep, the next state is chosen randomly.

The current state determines the probability distribution for the next state.

# A Markov System

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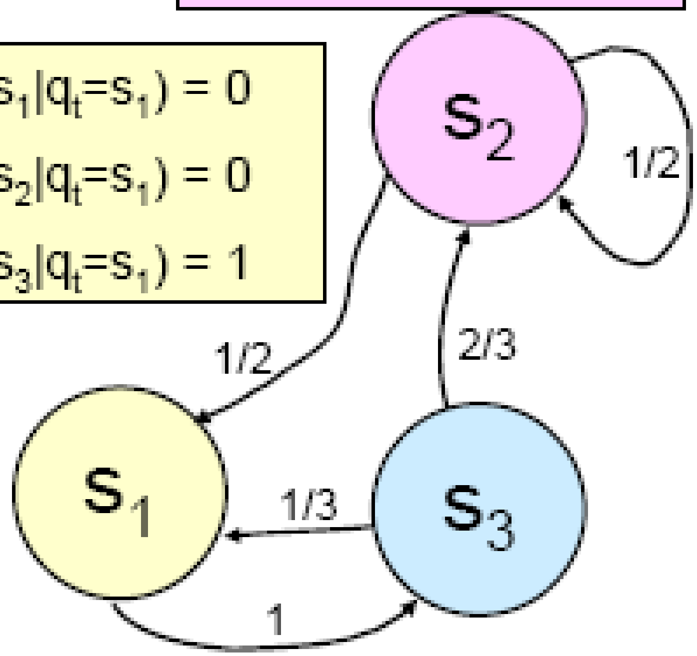
$P(q_{t+1}=s_3|q_t=s_1) = 1$




*N = 3*

*t=1*

$q_t=q_1=s_2$

$P(q_{t+1}=s_1|q_t=s_3) = 1/3$

$P(q_{t+1}=s_2|q_t=s_3) = 2/3$

$P(q_{t+1}=s_3|q_t=s_3) = 0$

**Often notated with arcs between states**

Has *N* states, called $s_1, s_2 .. s_N$

There are discrete timesteps, *t=0, t=1, …*

On the t'th timestep the system is in exactly one of the available states. Call it $q_t$

Note: $q_t \in \{s_1, s_2 .. s_N \}$

Between each timestep, the next state is chosen randomly.

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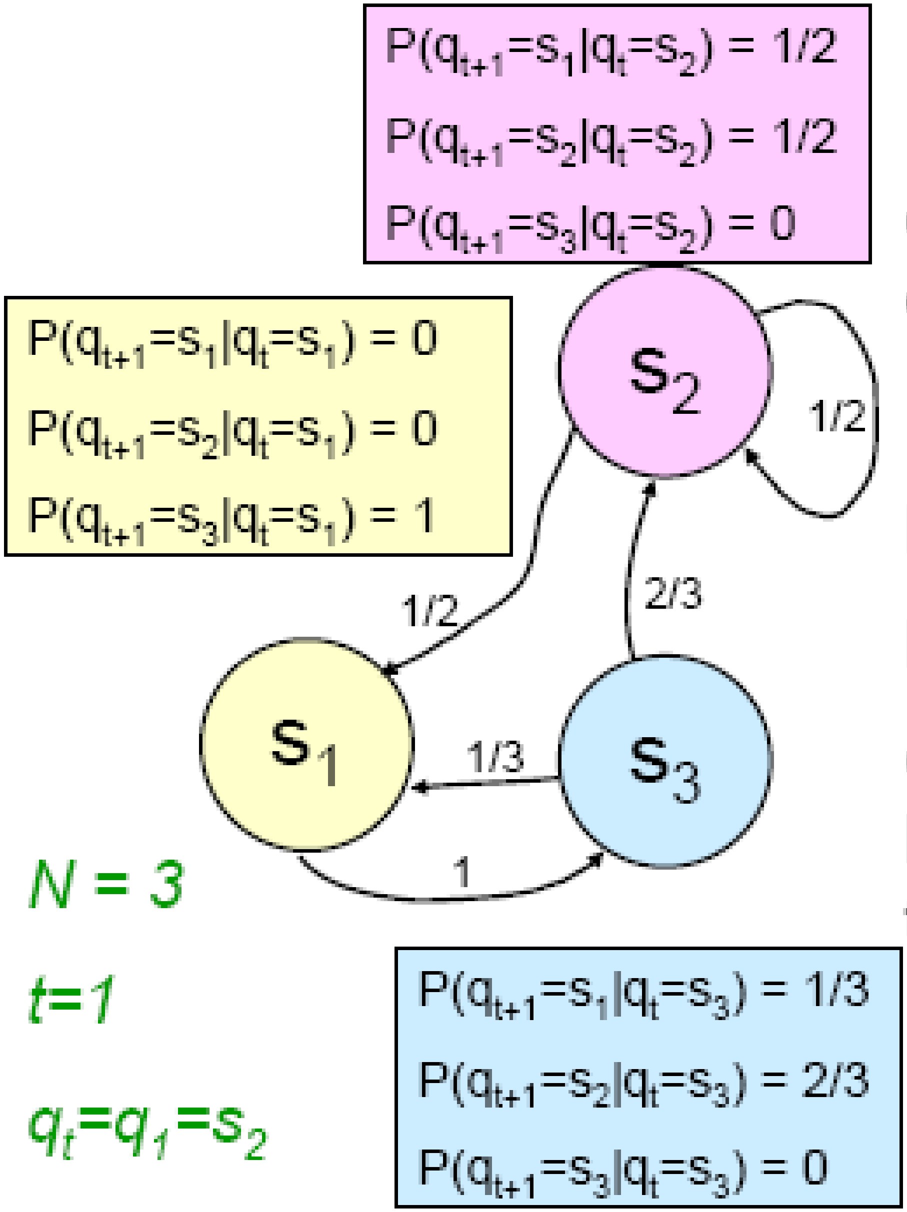
# Markov Property




*$N = 3$*

*$t=1$*

*$q_t=q_1=s_2$*

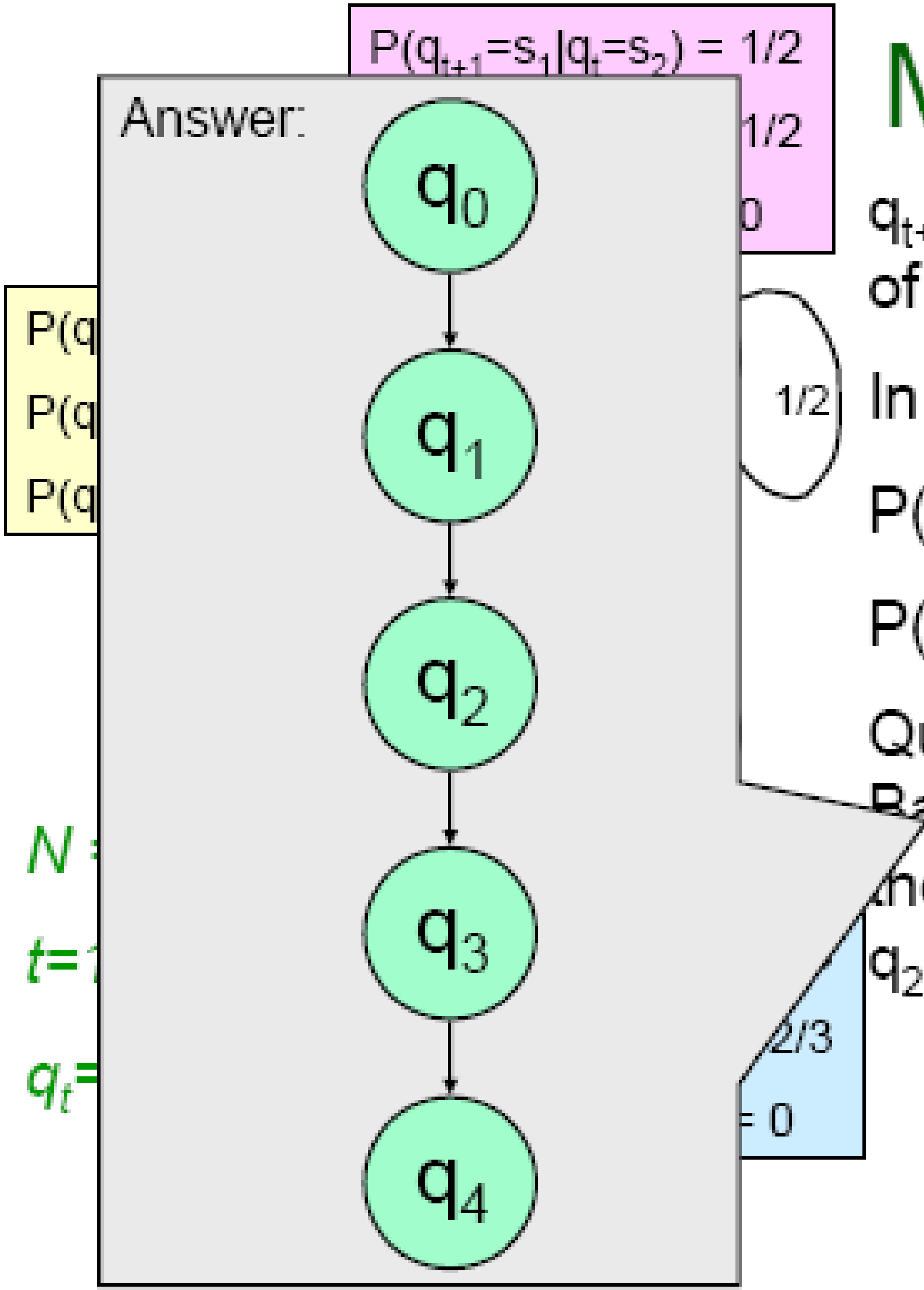
$q_{t+1}$ is conditionally independent of $\{ q_{t-1}, q_{t-2}, \ldots q_1, q_0 \}$ given $q_t$.

In other words:

$P(q_{t+1} = s_j | q_t = s_i) =$

$P(q_{t+1} = s_j | q_t = s_i$ ,any earlier history)

Question: what would be the best Bayes Net structure to represent the Joint Distribution of ( $q_0, q_1, \ldots q_3, q_4$ )?




# Markov Property

$q_{t+1}$ is conditionally independent of { $q_{t-1}$, $q_{t-2}$, … $q_1$, $q_0$ } given $q_t$.

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Question: what would be the best Bayes Net structure to represent the Joint Distribution of ( $q_0$, $q_1$, $q_2$,$q_3$,$q_4$ )?

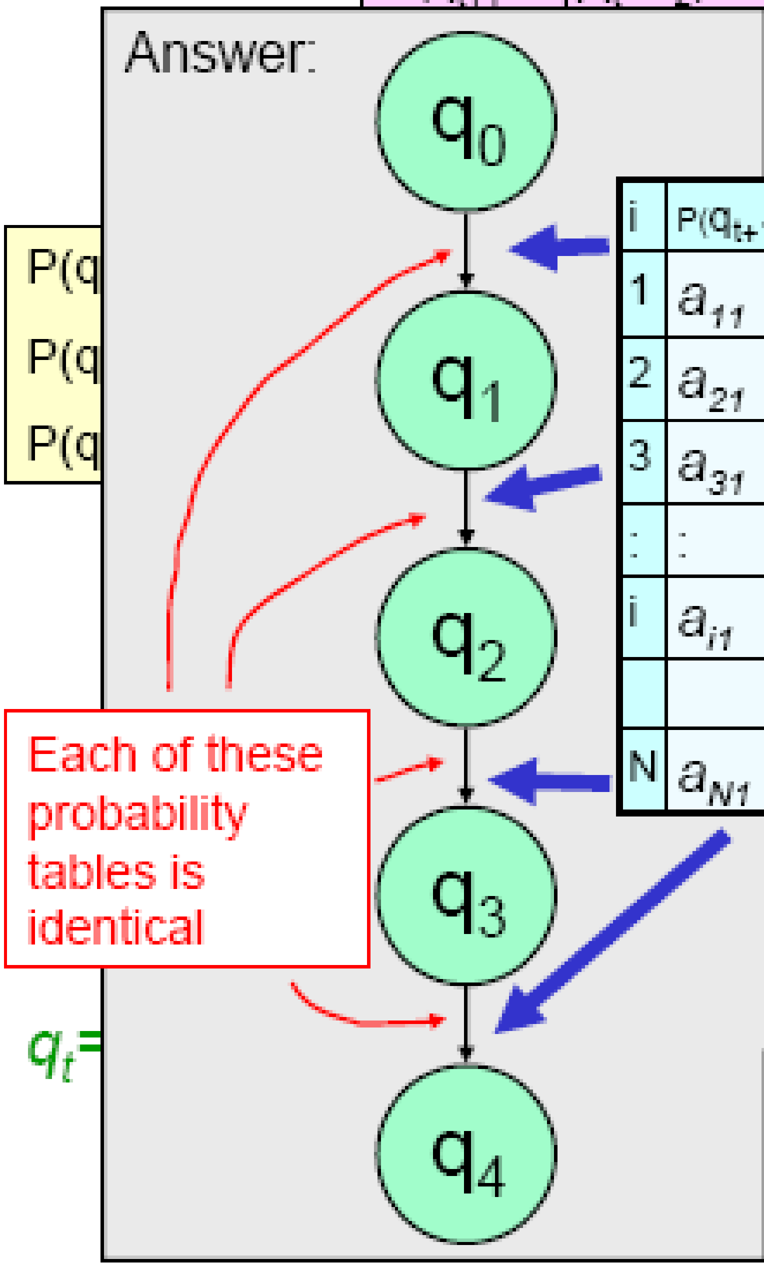
# Markov Property

$P(q_{t+1}=s_1|q_t=s_2) = 1/2$

1/2

0

$q_{t+1}$ is conditionally independent




| i | $P(q_{t+1}=s_1\|q_t=s_i)$ | $P(q_{t+1}=s_2\|q_t=s_i)$ | … | $P(q_{t+1}=s_j\|q_t=s_i)$ | … | $P(q_{t+1}=s_N\|q_t=s_i)$ |
|---|---|---|---|---|---|---|
| 1 | $a_{11}$ | $a_{12}$ | … | $a_{1j}$ | … | $a_{1N}$ |
| 2 | $a_{21}$ | $a_{22}$ | … | $a_{2j}$ | … | $a_{2N}$ |
| 3 | $a_{31}$ | $a_{32}$ | … | $a_{3j}$ | … | $a_{3N}$ |
| : | : | : | : | : | : | : |
| i | $a_{i1}$ | $a_{i2}$ | … | $a_{ij}$ | … | $a_{iN}$ |
| | | | | | | |
| N | $a_{N1}$ | $a_{N2}$ | … | $a_{Nj}$ | … | $a_{NN}$ |

the Joint Distribution of ( $q_0$, $q_1$, $q_2$,$q_3$,$q_4$ )?

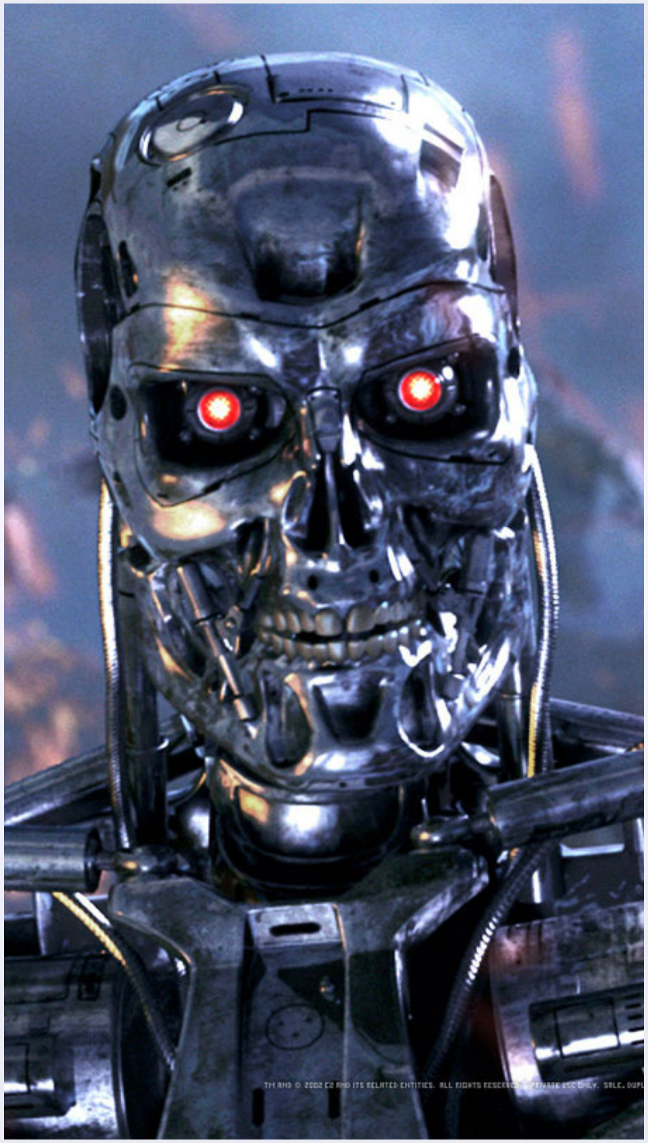
2/3

= 0

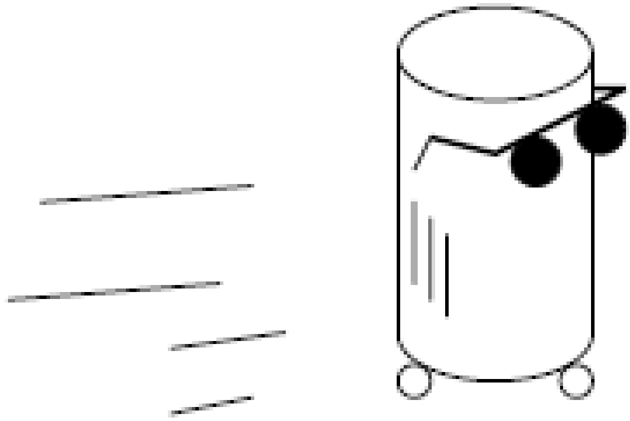
Notation:

$$a_{ij} = P(q_{t+1} = s_j \mid q_t = s_i)$$

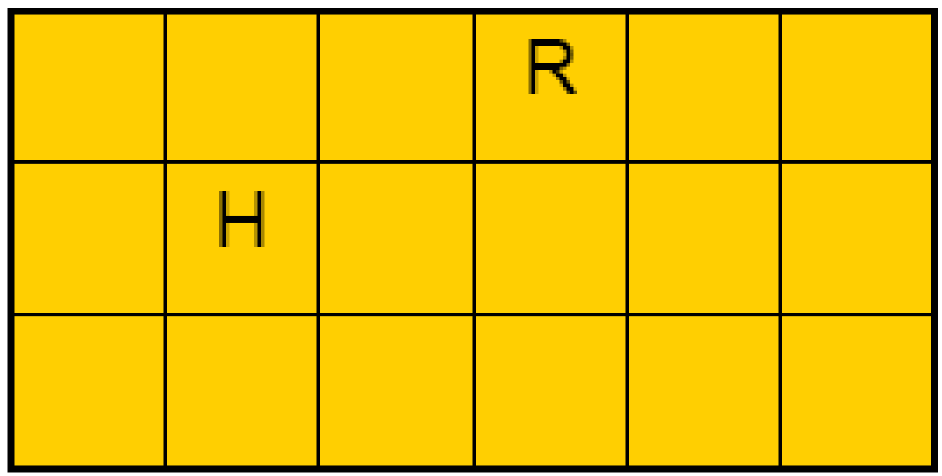
# Example



# A Blind Robot



A human and a robot wander around randomly on a grid…

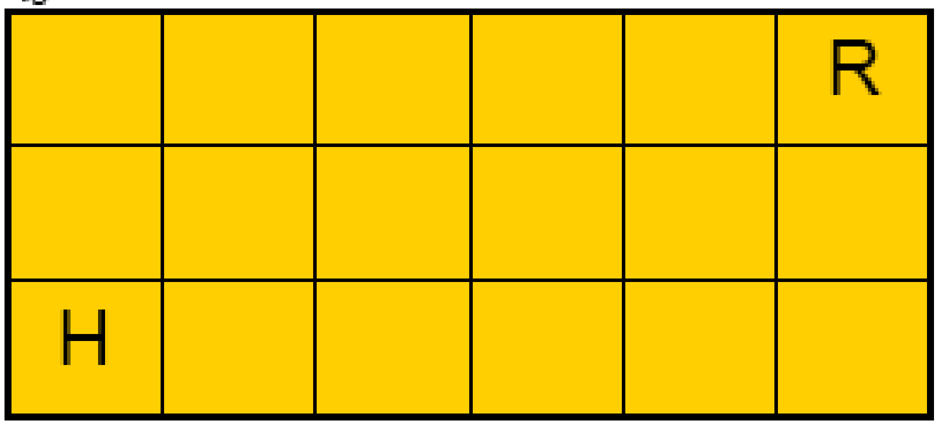



**STATE q =** ⟨ Location of Robot, Location of Human ⟩

Note: N (num. states) = 18 * 18 = 324

# Dynamics of System

$q_0$ =




Each timestep the human moves randomly to an adjacent cell. And Robot also moves randomly to an adjacent cell.

Typical Questions:

- "What's the expected time until the human is crushed like a bug?"
- "What's the probability that the robot will hit the left wall before it hits the human?"
- "What's the probability Robot crushes human on next time step?"

# Example Question

"It's currently time t, and human remains uncrushed. What's the probability of crushing occurring at time t + 1 ?"

If robot is blind:

We can compute this in advance.

We'll do this first

If robot is omnipotent:

(I.E. If robot knows state at time t), can compute directly.

Too Easy. We won't do this

If robot has some sensors, but incomplete state information …

Hidden Markov Models are applicable!

Main Body of Lecture

# What is $P(q_t = s)$? slow, stupid answer

Step 1: Work out how to compute $P(Q)$ for any path $Q$

$= q_1 q_2 q_3 .. q_t$

Given we know the start state $q_1$ (i.e. $P(q_1)=1$)

$$P(q_1 q_2 .. q_t) = P(q_1 q_2 .. q_{t-1}) P(q_t | q_1 q_2 .. q_{t-1})$$

$$= P(q_1 q_2 .. q_{t-1}) P(q_t | q_{t-1})$$ *WHY?*

$$= P(q_2|q_1)P(q_3|q_2)\ldots P(q_t|q_{t-1})$$

Step 2: Use this knowledge to get $P(q_t = s)$

$$P(q_t = s) = \sum_{Q \in \text{Paths of length } t \text{ that end in } s} P(Q)$$

Computation is exponential in t

# What is $P(q_t = s)$ ? Clever answer

- For each state $s_i$, define

  $p_t(i)$ = Prob. state is $s_i$ at time $t$

  $= P(q_t = s_i)$

- Easy to do inductive definition

$$\forall i \quad p_0(i) =$$

$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

# What is $P(q_t = s)$ ? Clever answer

- For each state $s_i$, define

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$$\forall i \quad p_0(i) = \begin{cases} 1 & \text{if } s_i \text{ is the start state} \\ 0 & \text{otherwise} \end{cases}$$

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$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

$$\sum_{i=1}^{N} P(q_{t+1} = s_j \wedge q_t = s_i) =$$

# What is $P(q_t = s)$ ? Clever answer

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$$\sum_{i=1}^{N} P(q_{t+1} = s_j \wedge q_t = s_i) =$$

$$\sum_{i=1}^{N} P(q_{t+1} = s_j \mid q_t = s_i) P(q_t = s_i) = \sum_{i=1}^{N} a_{ij} p_t(i)$$

Remember,

$$a_{ij} = P(q_{t+1} = s_j \mid q_t = s_i)$$

# What is $P(q_t = s)$ ? Clever answer

- For each state $s_i$, define

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$$\sum_{i=1}^{N} P(q_{t+1} = s_j \mid q_t = s_i) P(q_t = s_i) = \quad \sum_{i=1}^{N} a_{ij} p_t(i)$$

- Computation is simple.
- Just fill in this table in this order:

| $t$ | $p_t(1)$ | $p_t(2)$ | … | $p_t(N)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | | 0 |
| 1 | | | | |
| : | | | | |
| $t_{final}$ | | | | |

# What is $P(q_t = s)$ ? Clever answer

- For each state $s_i$, define

  $p_t(i)$ = Prob. state is $s_i$ at time $t$

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- Easy to do inductive definition

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$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

$$\sum_{i=1}^{N} P(q_{t+1} = s_j \wedge q_t = s_i) =$$

$$\sum_{i=1}^{N} P(q_{t+1} = s_j \mid q_t = s_i) P(q_t = s_i) = \sum_{i=1}^{N} a_{ij} p_t(i)$$

- Cost of computing $P_t(i)$ for all states $S_i$ is now $O(t N^2)$
- The stupid way was $O(N^t)$
- This was a simple example
- It was meant to warm you up to this trick, called *Dynamic Programming,* because HMMs do many tricks like this.

# Hidden State

"It's currently time t, and human remains uncrushed. What's the probability of crushing occurring at time t + 1 ?"

If robot is blind:

We can compute this in advance.

We'll do this first

If robot is omnipotent:

(I.E. If robot knows state at time t), can compute directly.

Too Easy. We won't do this

If robot has some sensors, but incomplete state information …

Hidden Markov Models are applicable!

Main Body of Lecture

# Hidden State

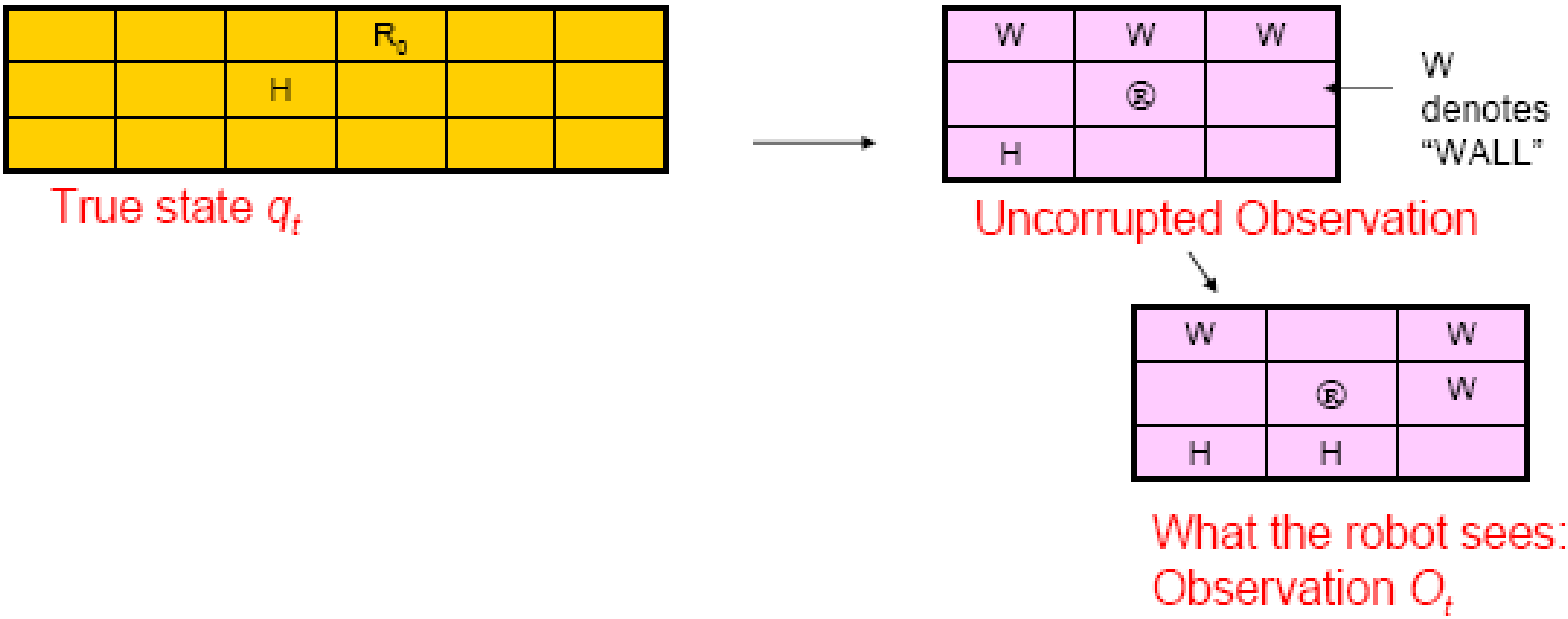
- The previous example tried to estimate $P(q_t = s_i)$ unconditionally (using no observed evidence).
- Suppose we can observe something that's affected by the true state.
- Example: Proximity sensors. (tell us the contents of the 8 adjacent squares)


True state $q_t$

What the robot sees: Observation $O_t$

# Noisy Hidden State

- Example: <u>Noisy Proximity sensors.</u> (unreliably tell us the contents of the 8 adjacent squares)



# Noisy Hidden State

- Example: Noisy Proximity sensors. (unreliable tell us the contents of the 8 adjacent squares)

| | | | | | |
|---|---|---|---|---|---|
| | | | $R_0$ | | 2 |
| | | H | | | |
| | | | | | |

True state $q_t$

→

| | | |
|---|---|---|
| W | W | W |
| | ® | |
| H | | |

← W denotes "WALL"

Uncorrupted Observation

↓

| | | |
|---|---|---|
| W | | W |
| | ® | W |
| H | H | |

What the robot sees: Observation $O_t$

$O_t$ is noisily determined depending on the current state.

Assume that $O_t$ is conditionally independent of $\{q_{t-1}, q_{t-2}, \ldots q_1, q_0, O_{t-1}, O_{t-2}, \ldots O_1, O_0\}$ given $q_t$.

In other words:

$P(O_t = X \mid q_t = s_i) =$

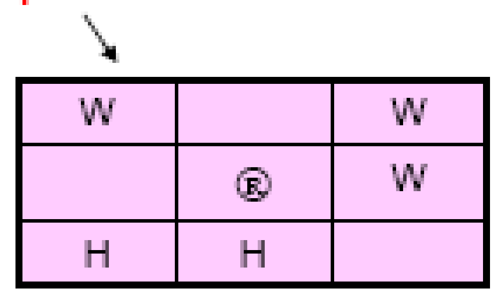
$P(O_t = X \mid q_t = s_i, \text{any earlier history})$

# Noisy Hidden State

- Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)


True state $q_t$

Uncorrupted Observation

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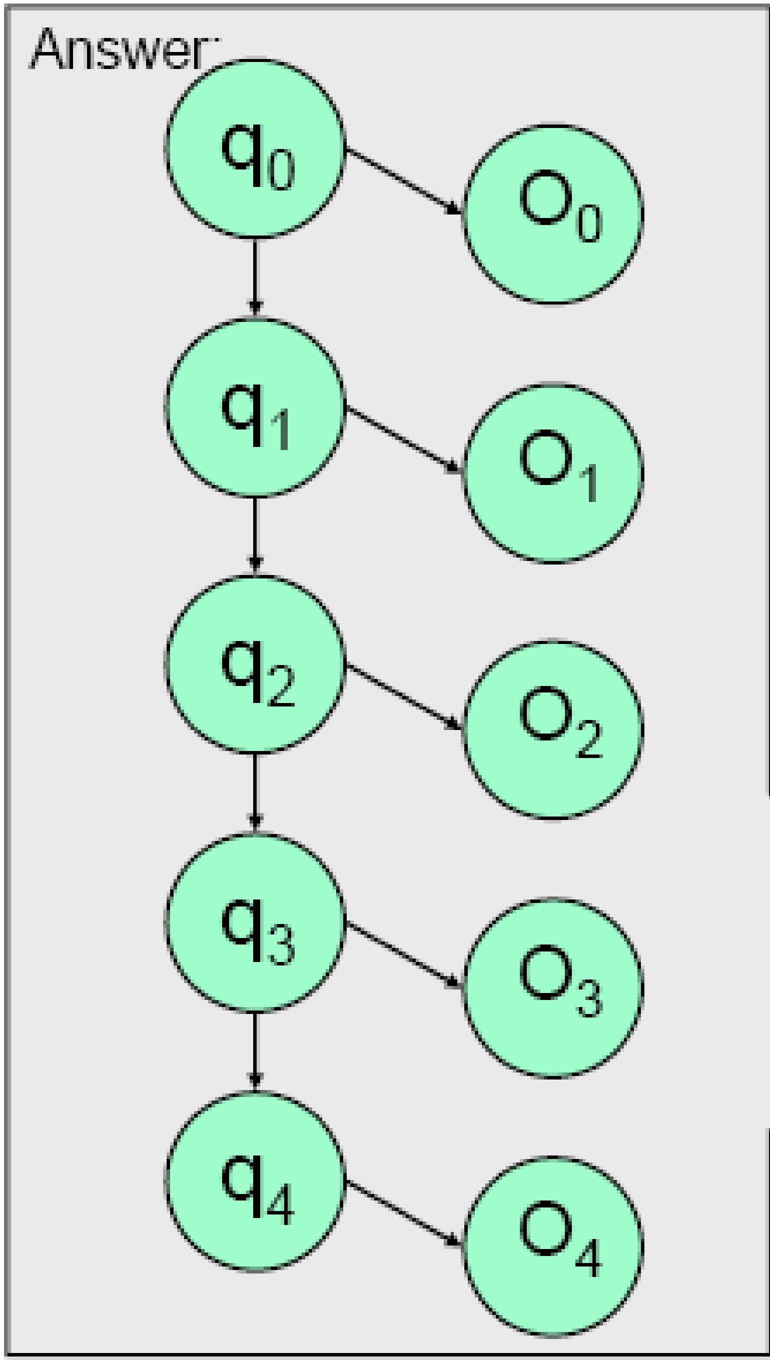
$P(O_t = X \mid q_t = s_i, \text{any earlier history})$

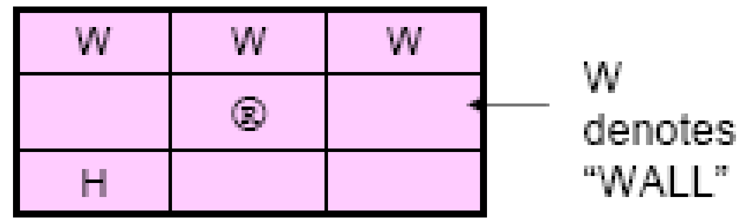
Question: what'd be the best Bayes Net structure to represent the Joint Distribution of $(q_0, q_1, q_2, q_3, q_4, O_0, O_1, O_2, O_3, O_4)$?

# Hidden State

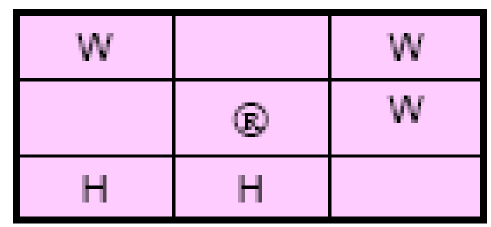
ximity sensors. (unreliably tell us

adjacent squares)




Uncorrupted Observation



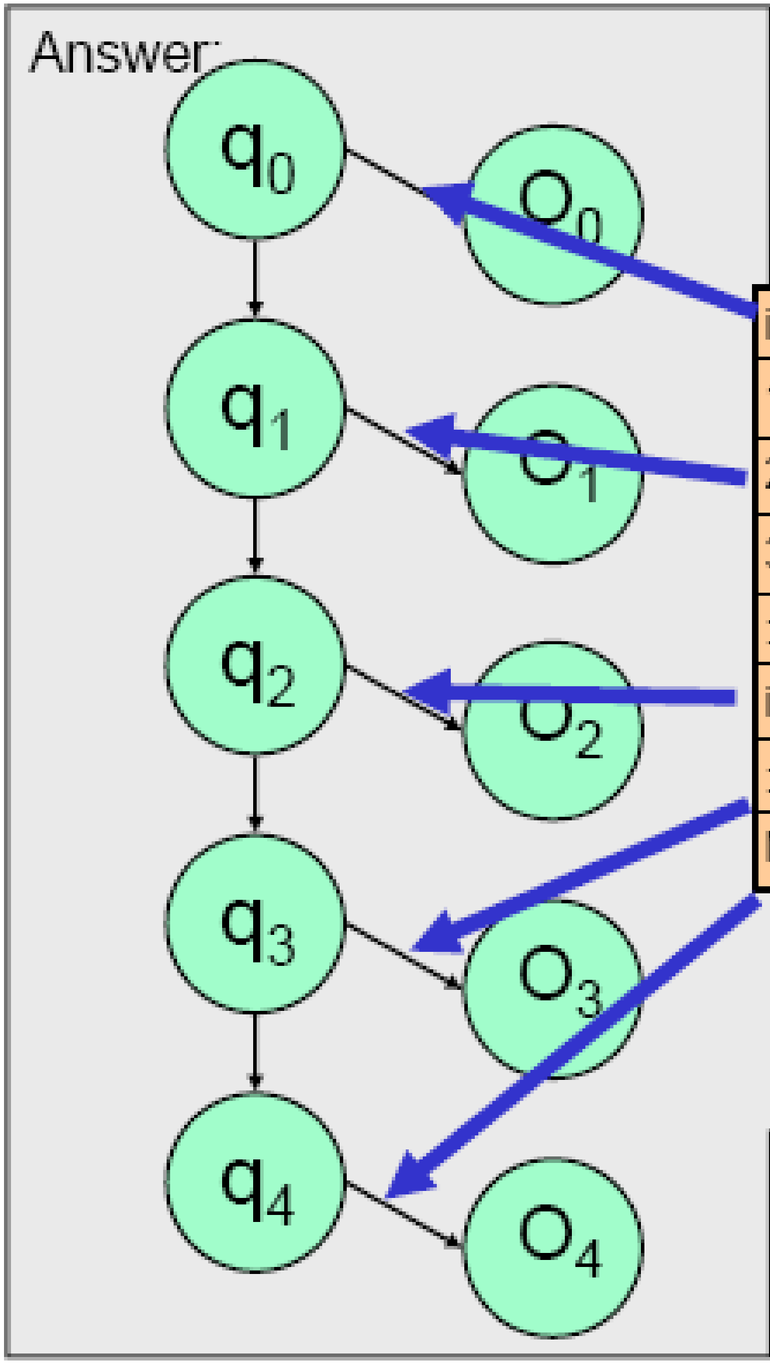

What the robot sees: Observation $O_t$

nding on

$O_{t-1}$,

story)

Question: what'd be the best Bayes Net structure to represent the Joint Distribution of ($q_0$, $q_1$, $q_2$,$q_3$,$q_4$ ,$O_0$, $O_1$, $O_2$,$O_3$,$O_4$ )?




# Hidden

ximity sens s

Notation:

$$b_i(k) = P(O_t = k \mid q_t = s_i)$$

| i | $P(O_t=1 \mid q_t=s_i)$ | $P(O_t=2 \mid q_t=s_i)$ | … | $P(O_t=k \mid q_t=s_i)$ | … | $P(O_t=M \mid q_t=s_i)$ |
|---|---|---|---|---|---|---|
| 1 | $b_1(1)$ | $b_1(2)$ | … | $b_1(k)$ | … | $b_1(M)$ |
| 2 | $b_2(1)$ | $b_2(2)$ | … | $b_2(k)$ | … | $b_2(M)$ |
| 3 | $b_3(1)$ | $b_3(2)$ | … | $b_3(k)$ | … | $b_3(M)$ |
| : | : | : | : | : | : | : |
| i | $b_i(1)$ | $b_i(2)$ | … | $b_i(k)$ | … | $b_i(M)$ |
| : | : | : | : | : | : | : |
| N | $b_N(1)$ | $b_N(2)$ | … | $b_N(k)$ | … | $b_N(M)$ |

$O_{t-1}$,

| H | H | |

What the robot sees: Observation $O_t$

story)

Question: what'd be the best Bayes Net structure to represent the Joint Distribution of ($q_0$, $q_1$, $q_2$, $q_3$, $q_4$, $O_0$, $O_1$, $O_2$, $O_3$, $O_4$)?

# Hidden Markov Models

Our robot with noisy sensors is a good example of an HMM

- Question 1: State Estimation

  What is $P(q_T=S_i \mid O_1 O_2 \ldots O_T)$

  It will turn out that a new cute D.P. trick will get this for us.

- Question 2: Most Probable Path

  Given $O_1 O_2 \ldots O_T$ , what is the most probable path that I took?

  And what is that probability?

  Yet another famous D.P. trick, the VITERBI algorithm, gets this.

- Question 3: Learning HMMs:

  Given $O_1 O_2 \ldots O_T$ , what is the maximum likelihood HMM that could have produced this string of observations?

  Very very useful. Uses the E.M. Algorithm

# Some Famous HMM Tasks

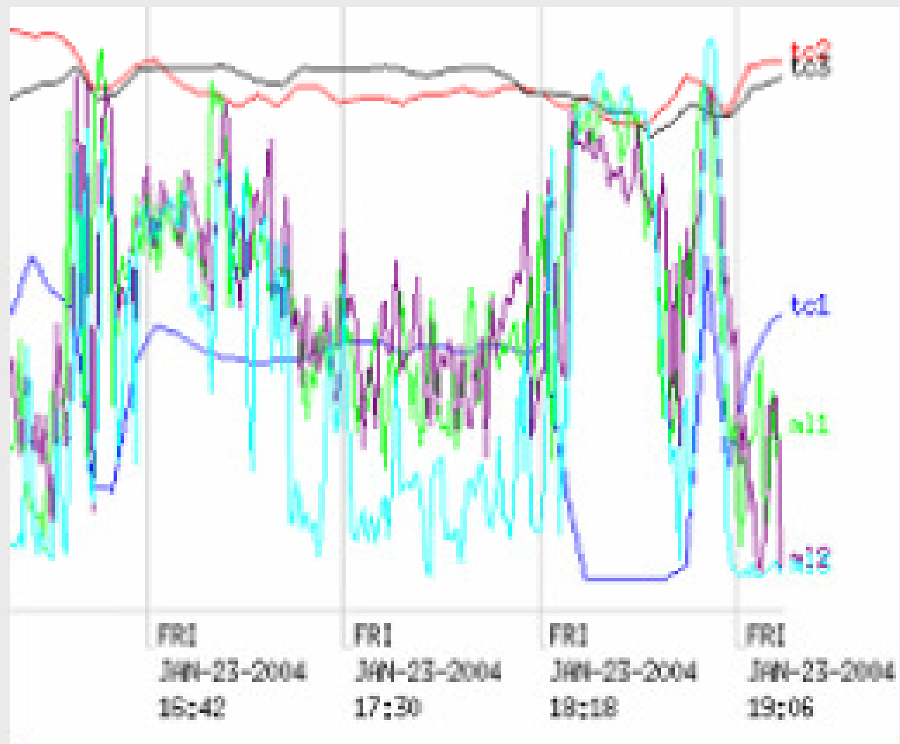
Question 1: State Estimation

What is $P(q_T=S_i \mid O_1O_2 \ldots O_t)$

# Some Famous HMM Tasks

Question 1: State Estimation

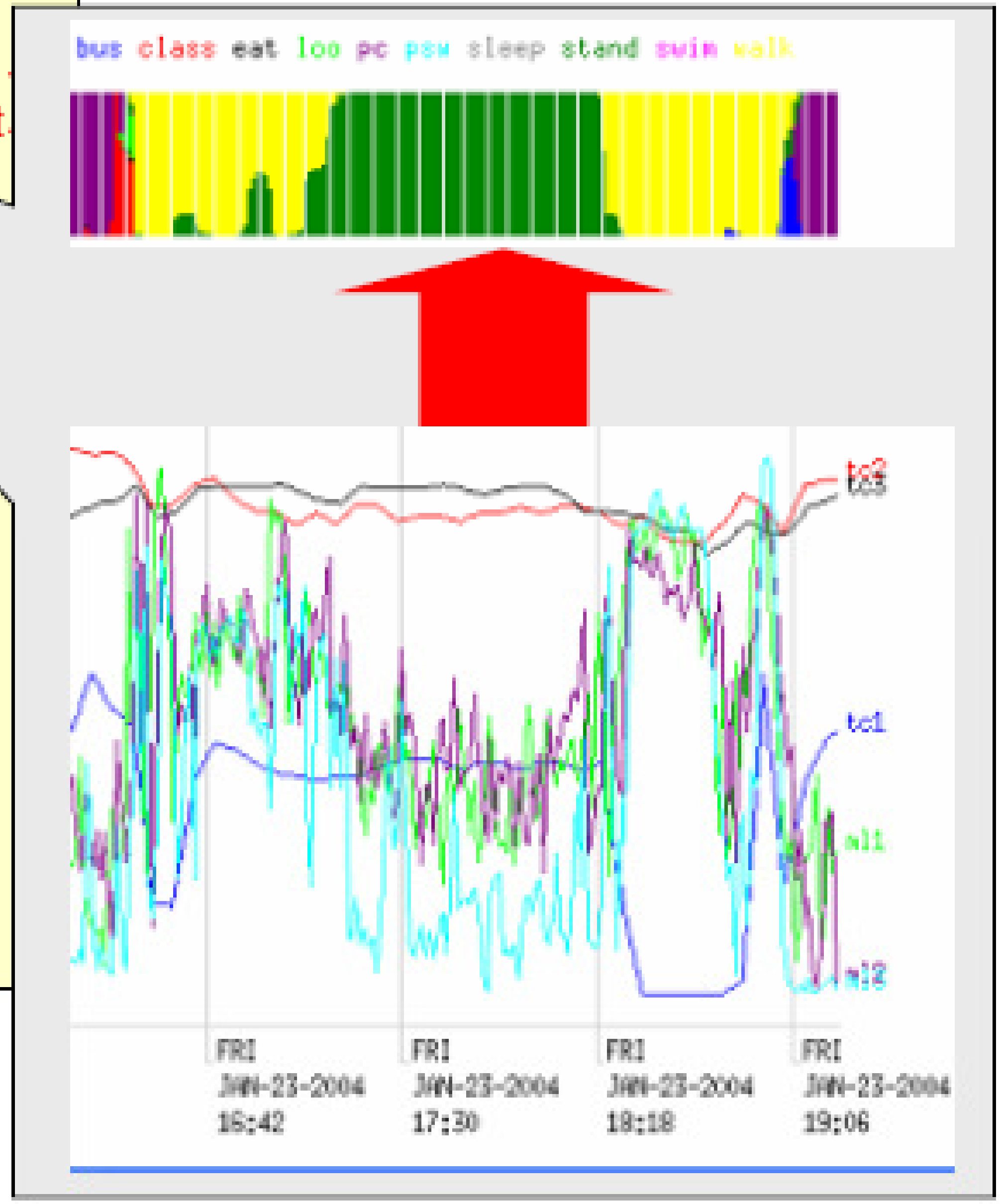
What is $P(q_T=S_i | O_1O_2\ldots O_t)$



# Some Famous HMM Tasks

**Question 1: State Estimation**

What is $P(q_T=S_i \mid O_1O_2\ldots O_t)$



# Some Famous HMM Tasks

Question 1: State Estimation

What is $P(q_T=S_i \mid O_1O_2\ldots O_t)$

Question 2: Most Probable Path

Given $O_1O_2\ldots O_T$ , what is the most probable path that I took?

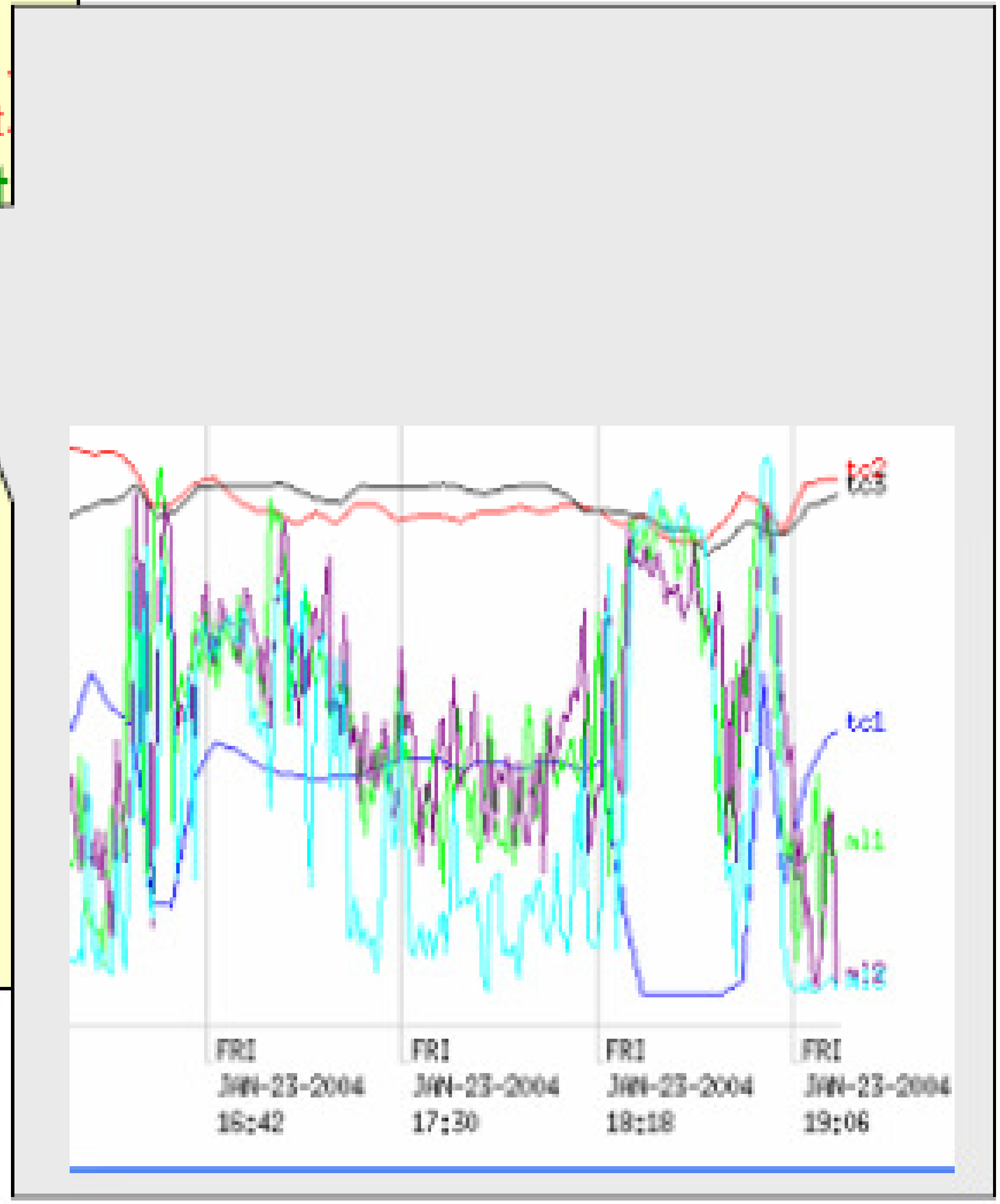
# Some Famous HMM Tasks

Question 1: State Estimation

What is $P(q_T=S_i \mid O_1O_2\ldots O_t$

Question 2: Most Probable Pat

Given $O_1O_2\ldots O_T$ , what is the most probable path that I took?



# Some Famous HMM Tasks

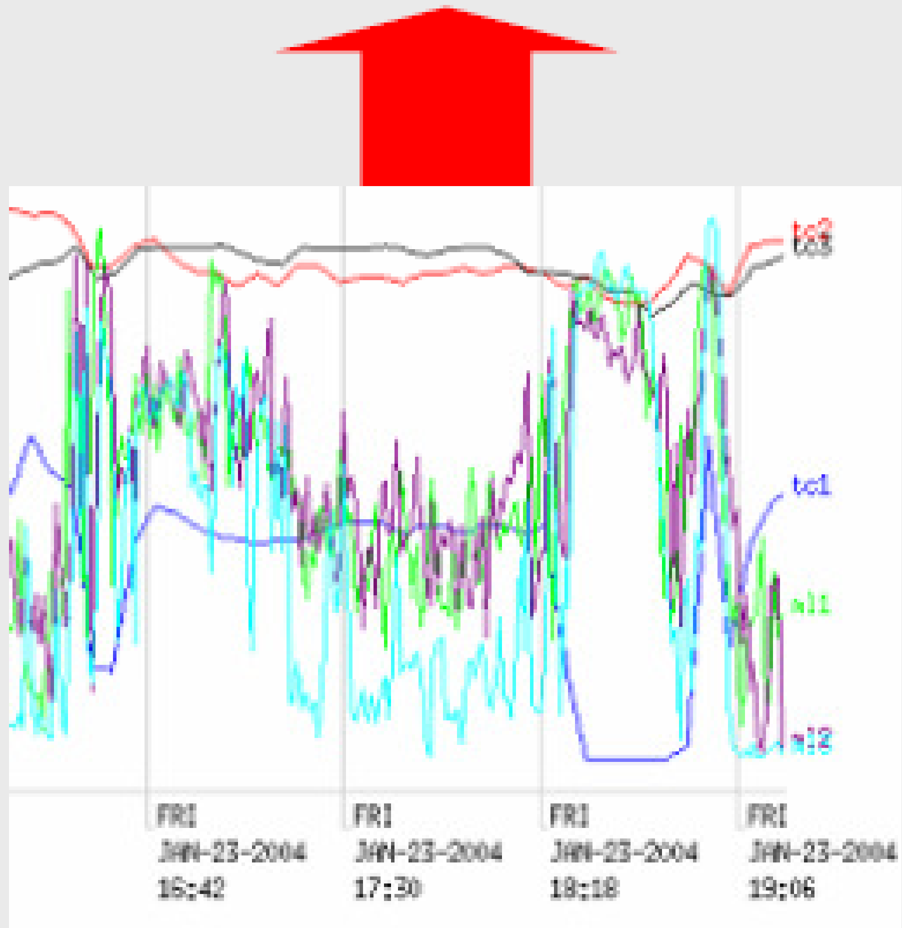
Question 1: State Estimation

What is $P(q_T=S_i \mid O_1O_2\ldots O_t$

Question 2: Most Probable Pat

Given $O_1O_2\ldots O_T$ , what is the most probable path that I took?

Woke up at 8.35, Got on Bus at 9.46, Sat in lecture 10.05-11.22…



# Some Famous HMM Tasks

Question 1: State Estimation

What is $P(q_T=S_i \mid O_1O_2\ldots O_t)$

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Given $O_1O_2\ldots O_T$, what is the maximum likelihood HMM that could have produced this string of observations?

# Some Famou

Question 1: State Estimation

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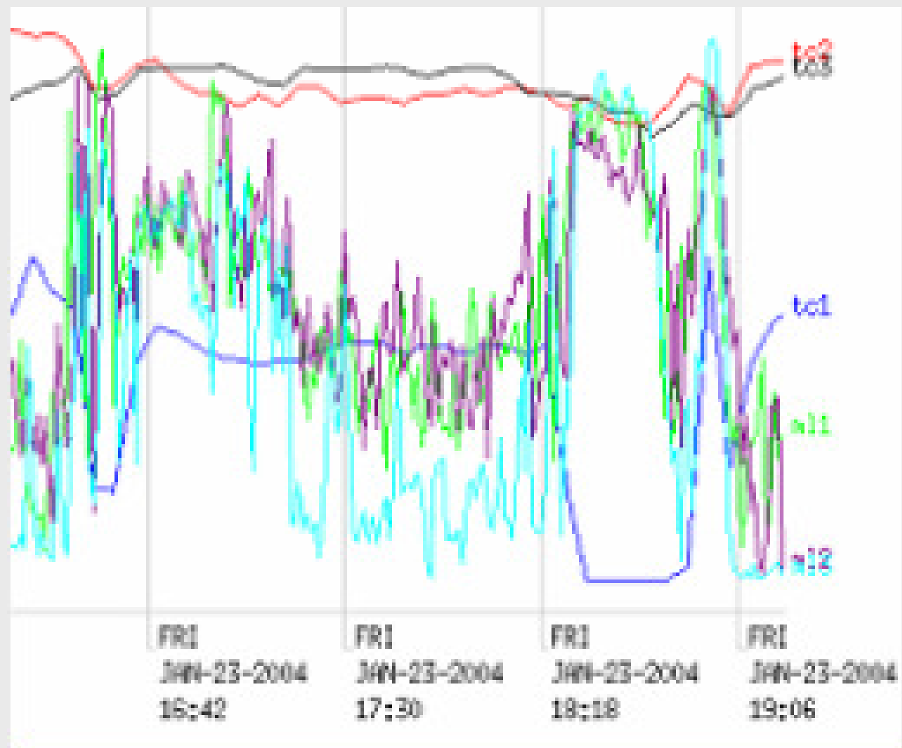

# Some Famou

Question 1: State Estimation

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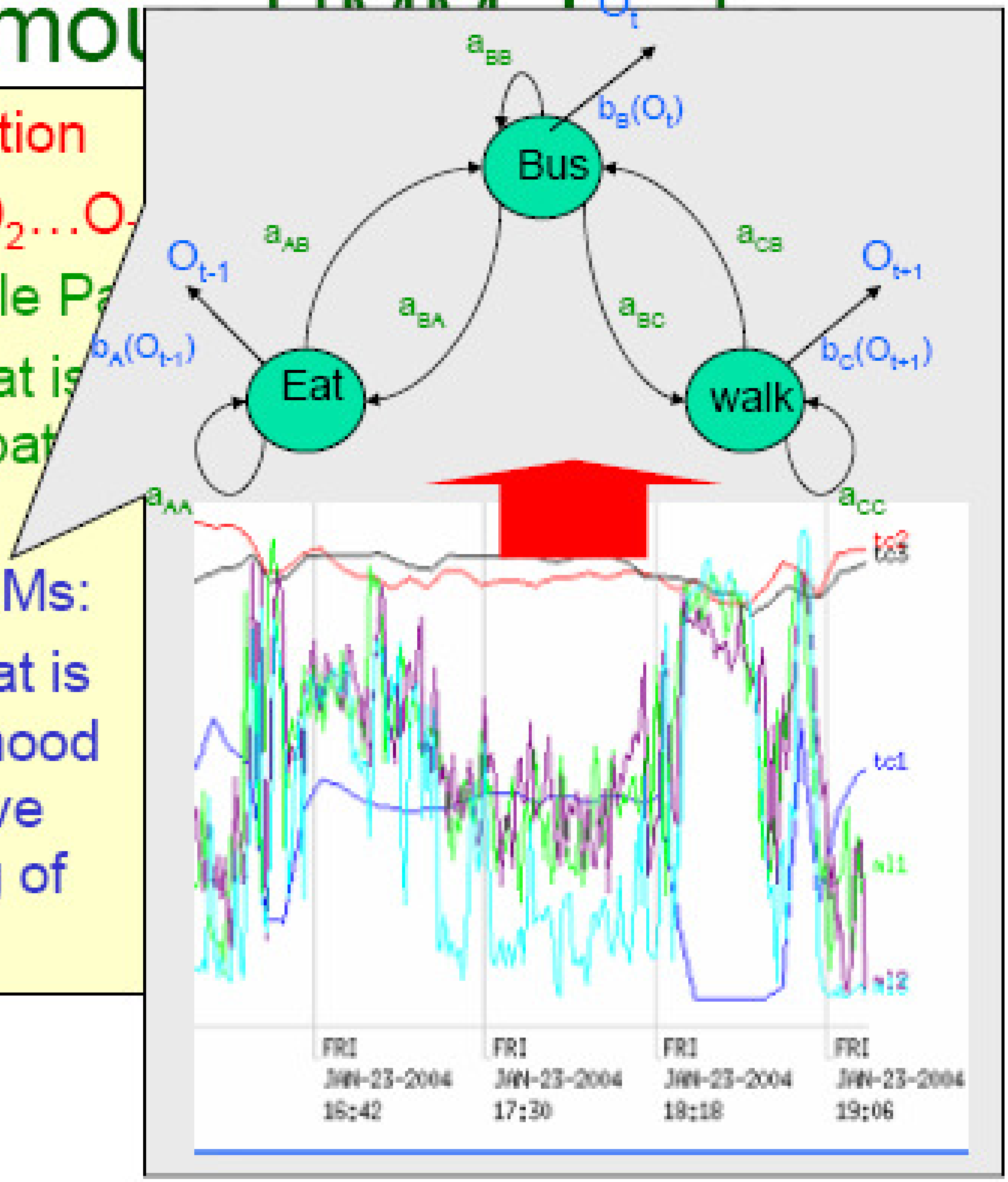
Given $O_1O_2\ldots O_T$, what is the maximum likelihood HMM that could have produced this string of observations?



# Basic Operations in HMMs

For an observation sequence $O = O_1 \ldots O_T$, the three basic HMM operations are:

| Problem | Algorithm | Complexity |
|---|---|---|
| *Evaluation:* Calculating $P(q_t=S_i \mid O_1O_2\ldots O_t)$ | Forward-Backward | $O(TN^2)$ |
| *Inference:* Computing $Q^* = argmax_Q\, P(Q\|O)$ | Viterbi Decoding | $O(TN^2)$ |
| *Learning:* Computing $\lambda^* = argmax_\lambda\, P(O\|\lambda)$ | Baum-Welch (EM) | $O(TN^2)$ |

T = # timesteps, N = # states

# HMM Notation (from Rabiner's Survey)*

*L. R. Rabiner, "A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition," Proc. of the IEEE, Vol.77, No.2, pp.257--286, 1989.

Available from http://ieeexplore.ieee.org/iel5/5/698/00018626.pdf?arnumber=18626

The states are labeled $S_1\ S_2\ ..\ S_N$

For a particular trial….

Let $T$ be the number of observations

$T$ is also the number of states passed through

$O = O_1\ O_2\ ..\ O_T$ is the sequence of observations

$Q = q_1\ q_2\ ..\ q_T$ is the notation for a path of states

$\lambda = \langle N, M, \{\pi_i\}, \{a_{ij}\}, \{b_i(j)\} \rangle$ is the specification of an HMM

# HMM Formal Definition

An HMM, $\lambda$, is a 5-tuple consisting of

- N the number of states
- M the number of possible observations
- $\{\pi_1, \pi_2, .. \pi_N\}$ The starting state probabilities

  $P(q_0 = S_i) = \pi_i$

This is new. In our previous example, start state was deterministic

- $a_{11}$ $a_{22}$ … $a_{1N}$

  $a_{21}$ $a_{22}$ … $a_{2N}$

  : : :

  $a_{N1}$ $a_{N2}$ … $a_{NN}$

  The state transition probabilities

  $P(q_{t+1}=S_j \mid q_t=S_i)=a_{ij}$

- $b_1(1)$ $b_1(2)$ … $b_1(M)$

  $b_2(1)$ $b_2(2)$ … $b_2(M)$

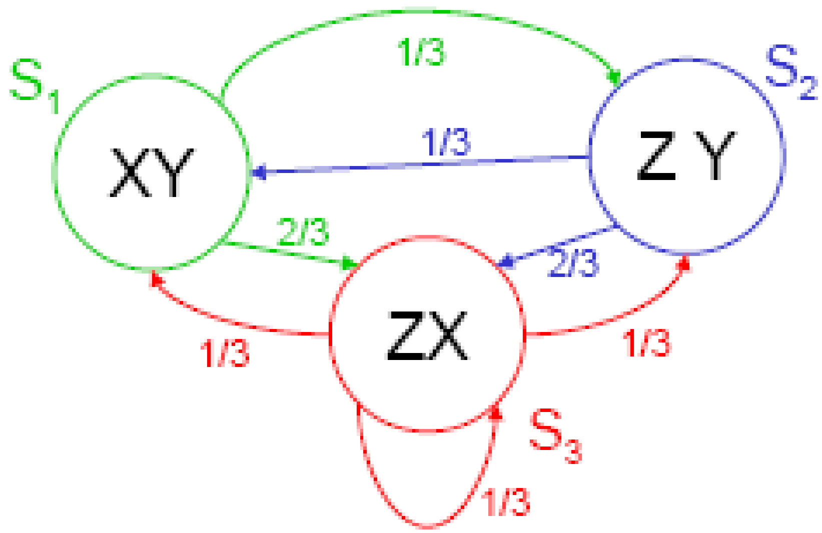
  : : :

  $b_N(1)$ $b_N(2)$ … $b_N(M)$

  The observation probabilities

  $P(O_t=k \mid q_t=S_i)=b_i(k)$

# Here's an HMM




Start randomly in state 1 or 2

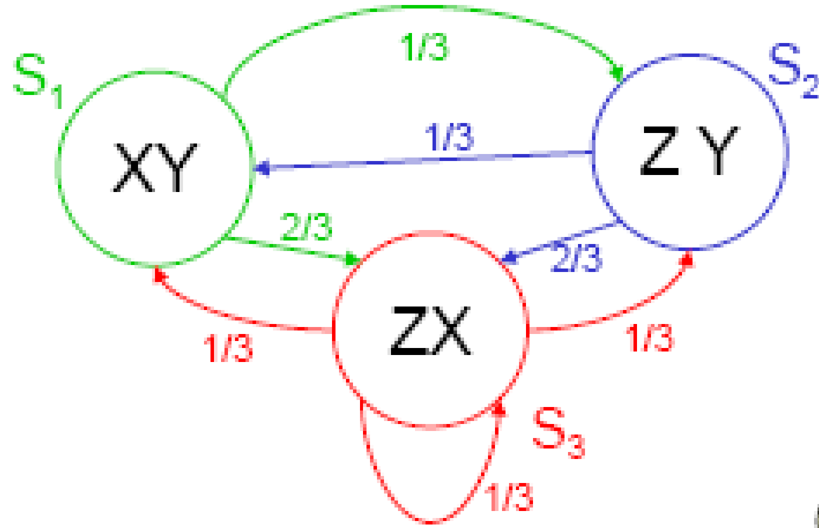
Choose one of the output symbols in each state at random.

N = 3

M = 3

| | | |
|---|---|---|
| $\pi_1 = 1/2$ | $\pi_2 = 1/2$ | $\pi_3 = 0$ |
| $a_{11} = 0$ | $a_{12} = 1/3$ | $a_{13} = 2/3$ |
| $a_{12} = 1/3$ | $a_{22} = 0$ | $a_{13} = 2/3$ |
| $a_{13} = 1/3$ | $a_{32} = 1/3$ | $a_{13} = 1/3$ |
| $b_1(X) = 1/2$ | $b_1(Y) = 1/2$ | $b_1(Z) = 0$ |
| $b_2(X) = 0$ | $b_2(Y) = 1/2$ | $b_2(Z) = 1/2$ |
| $b_3(X) = 1/2$ | $b_3(Y) = 0$ | $b_3(Z) = 1/2$ |

# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

50-50 choice between $S_1$ and $S_2$

| | | | |
|---|---|---|---|
| $q_0$= | __ | $O_0$= | __ |
| $q_1$= | __ | $O_1$= | __ |
| $q_2$= | __ | $O_2$= | __ |

N = 3

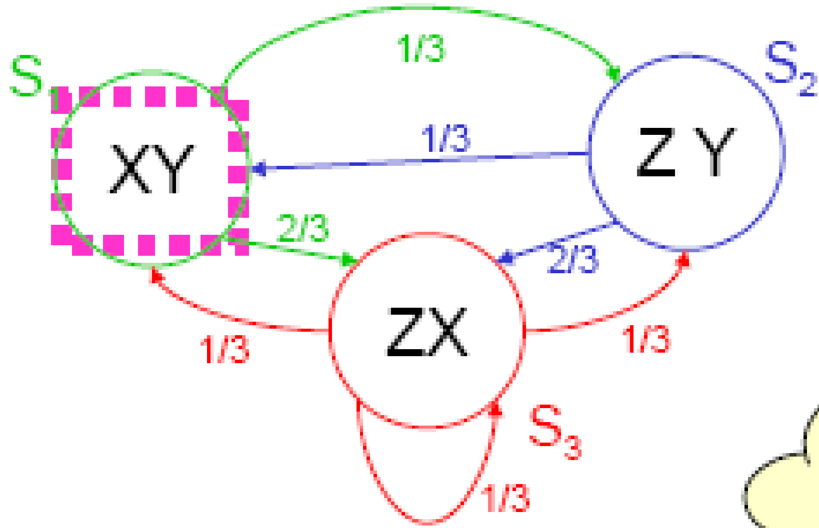
M = 3

$\pi_1 = ½$ $\pi_2 = ½$ $\pi_3 = 0$

| | | |
|---|---|---|
| $a_{11} = 0$ | $a_{12} = ⅓$ | $a_{13} = ⅔$ |
| $a_{12} = ⅓$ | $a_{22} = 0$ | $a_{13} = ⅔$ |
| $a_{13} = ⅓$ | $a_{32} = ⅓$ | $a_{13} = ⅓$ |

| | | |
|---|---|---|
| $b_1$ (X) = ½ | $b_1$ (Y) = ½ | $b_1$ (Z) = 0 |
| $b_2$ (X) = 0 | $b_2$ (Y) = ½ | $b_2$ (Z) = ½ |
| $b_3$ (X) = ½ | $b_3$ (Y) = 0 | $b_3$ (Z) = ½ |

# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

50-50 choice between X and Y

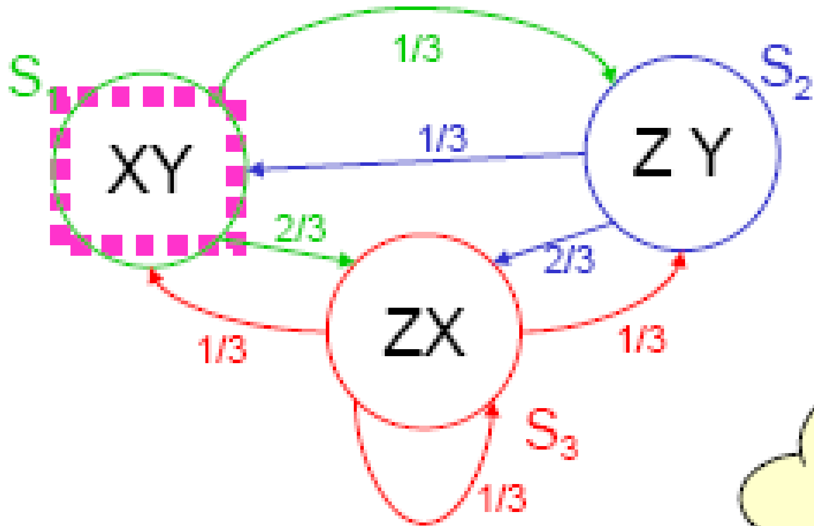
$N = 3$

$M = 3$

| | | |
|---|---|---|
| $\pi_1 = ½$ | $\pi_2 = ½$ | $\pi_3 = 0$ |
| | | |
| $a_{11} = 0$ | $a_{12} = ⅓$ | $a_{13} = ⅔$ |
| $a_{12} = ⅓$ | $a_{22} = 0$ | $a_{13} = ⅔$ |
| $a_{13} = ⅓$ | $a_{32} = ⅓$ | $a_{13} = ⅓$ |
| | | |
| $b_1$ (X) = ½ | $b_1$ (Y) = ½ | $b_1$ (Z) = 0 |
| $b_2$ (X) = 0 | $b_2$ (Y) = ½ | $b_2$ (Z) = ½ |
| $b_3$ (X) = ½ | $b_3$ (Y) = 0 | $b_3$ (Z) = ½ |

| | | | |
|---|---|---|---|
| $q_0=$ | $S_1$ | $O_0=$ | __ |
| $q_1=$ | __ | $O_1=$ | __ |
| $q_2=$ | __ | $O_2=$ | __ |

# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

Goto $S_3$ with probability 2/3 or $S_2$ with prob. 1/3

$N = 3$

$M = 3$

$\pi_1 = ½$ $\pi_2 = ½$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = ⅓$ $a_{13} = ⅔$

$a_{12} = ⅓$ $a_{22} = 0$ $a_{13} = ⅔$

$a_{13} = ⅓$ $a_{32} = ⅓$ $a_{13} = ⅓$

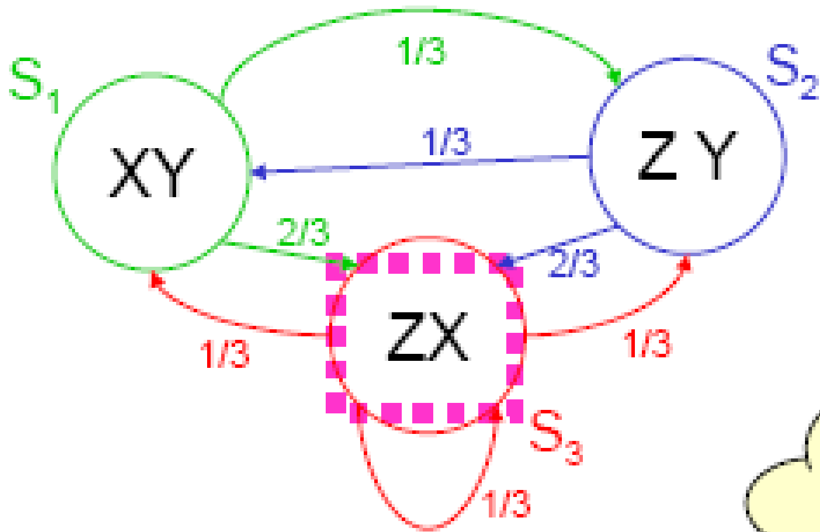
$b_1 (X) = ½$ $b_1 (Y) = ½$ $b_1 (Z) = 0$

$b_2 (X) = 0$ $b_2 (Y) = ½$ $b_2 (Z) = ½$

$b_3 (X) = ½$ $b_3 (Y) = 0$ $b_3 (Z) = ½$

| | | | |
|---|---|---|---|
| $q_0=$ | $S_1$ | $O_0=$ | X |
| $q_1=$ | __ | $O_1=$ | __ |
| $q_2=$ | __ | $O_2=$ | __ |

# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

50-50 choice between Z and X

$N = 3$

$M = 3$

$\pi_1 = ½$ $\pi_2 = ½$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = ⅓$ $a_{13} = ⅔$

$a_{12} = ⅓$ $a_{22} = 0$ $a_{13} = ⅔$

$a_{13} = ⅓$ $a_{32} = ⅓$ $a_{13} = ⅓$

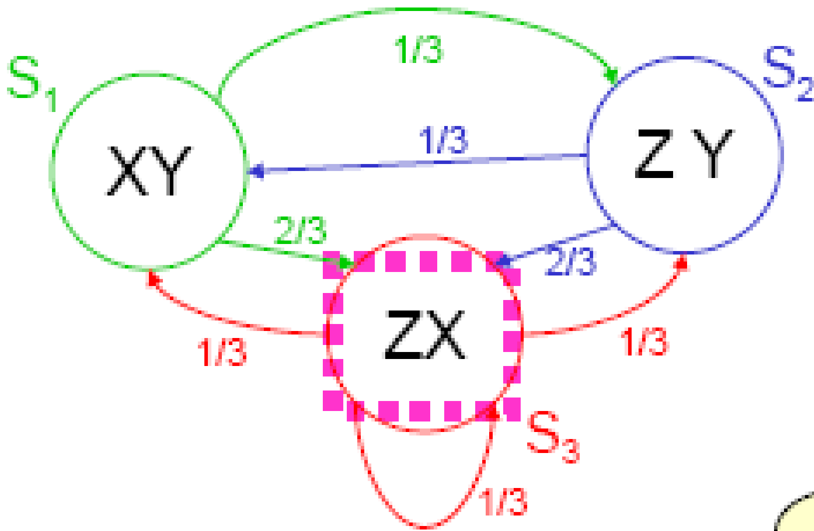
$b_1(X) = ½$ $b_1(Y) = ½$ $b_1(Z) = 0$

$b_2(X) = 0$ $b_2(Y) = ½$ $b_2(Z) = ½$

$b_3(X) = ½$ $b_3(Y) = 0$ $b_3(Z) = ½$

| | | | |
|---|---|---|---|
| $q_0=$ | $S_1$ | $O_0=$ | X |
| $q_1=$ | $S_3$ | $O_1=$ | __ |
| $q_2=$ | __ | $O_2=$ | __ |

# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

Each of the three next states is equally likely

$N = 3$

$M = 3$

$\pi_1 = ½$ $\pi_2 = ½$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = ⅓$ $a_{13} = ⅔$

$a_{12} = ⅓$ $a_{22} = 0$ $a_{13} = ⅔$

$a_{13} = ⅓$ $a_{32} = ⅓$ $a_{13} = ⅓$

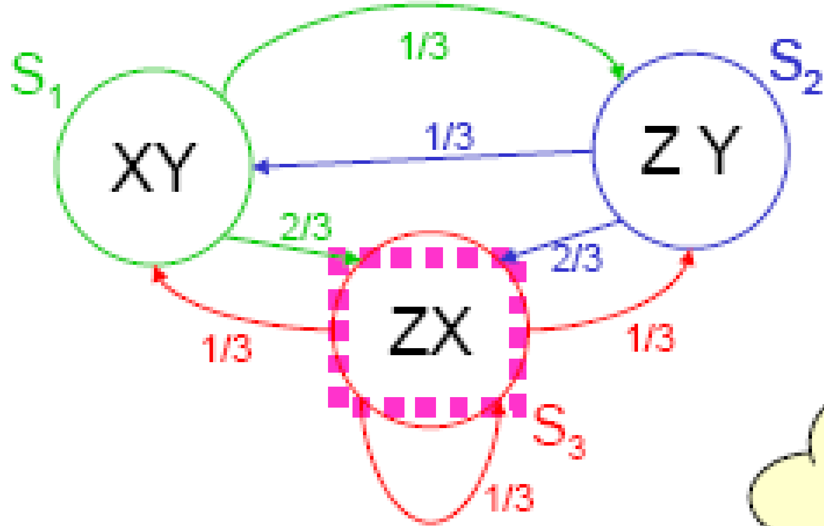
$b_1(X) = ½$ $b_1(Y) = ½$ $b_1(Z) = 0$

$b_2(X) = 0$ $b_2(Y) = ½$ $b_2(Z) = ½$

$b_3(X) = ½$ $b_3(Y) = 0$ $b_3(Z) = ½$

| | | | |
|---|---|---|---|
| $q_0$= | $S_1$ | $O_0$= | X |
| $q_1$= | $S_3$ | $O_1$= | X |
| $q_2$= | __ | $O_2$= | __ |

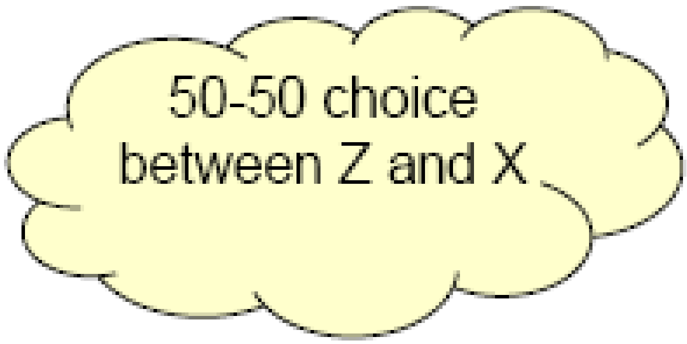
# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:



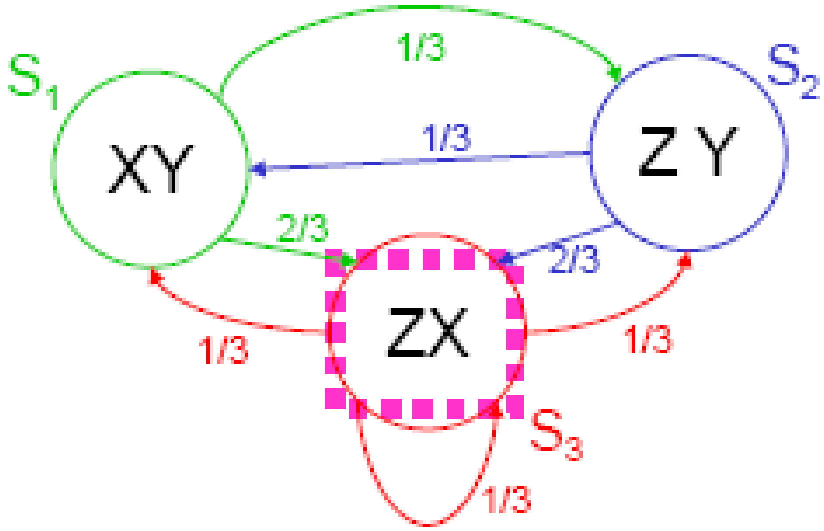

$N = 3$

$M = 3$

| | | |
|---|---|---|
| $\pi_1 = ½$ | $\pi_2 = ½$ | $\pi_3 = 0$ |
| $a_{11} = 0$ | $a_{12} = ⅓$ | $a_{13} = ⅔$ |
| $a_{12} = ⅓$ | $a_{22} = 0$ | $a_{13} = ⅔$ |
| $a_{13} = ⅓$ | $a_{32} = ⅓$ | $a_{13} = ⅓$ |
| $b_1(X) = ½$ | $b_1(Y) = ½$ | $b_1(Z) = 0$ |
| $b_2(X) = 0$ | $b_2(Y) = ½$ | $b_2(Z) = ½$ |
| $b_3(X) = ½$ | $b_3(Y) = 0$ | $b_3(Z) = ½$ |

| | | | |
|---|---|---|---|
| $q_0=$ | $S_1$ | $O_0=$ | X |
| $q_1=$ | $S_3$ | $O_1=$ | X |
| $q_2=$ | $S_3$ | $O_2=$ | __ |

# Here's an HMM




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

N = 3

M = 3

$\pi_1 = ½$ $\pi_2 = ½$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = ⅓$ $a_{13} = ⅔$

$a_{12} = ⅓$ $a_{22} = 0$ $a_{13} = ⅔$

$a_{13} = ⅓$ $a_{32} = ⅓$ $a_{13} = ⅓$

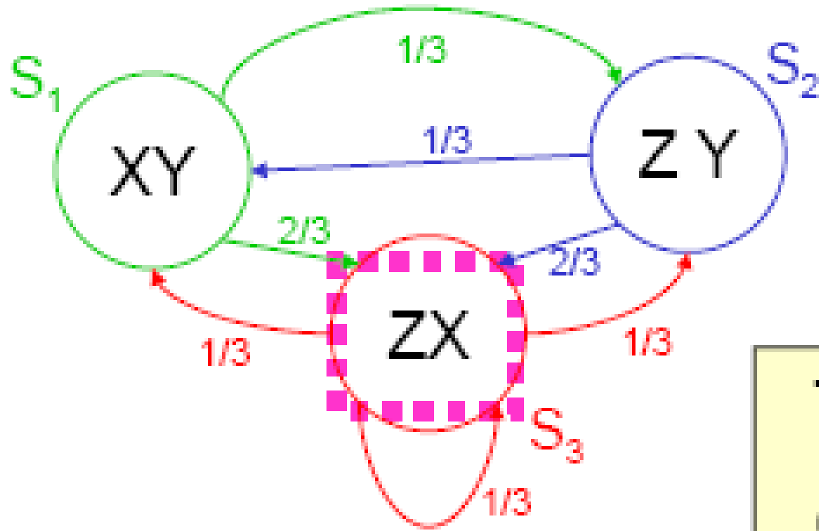
$b_1(X) = ½$ $b_1(Y) = ½$ $b_1(Z) = 0$

$b_2(X) = 0$ $b_2(Y) = ½$ $b_2(Z) = ½$

$b_3(X) = ½$ $b_3(Y) = 0$ $b_3(Z) = ½$

| | | | |
|---|---|---|---|
| $q_0$= | $S_1$ | $O_0$= | X |
| $q_1$= | $S_3$ | $O_1$= | X |
| $q_2$= | $S_3$ | $O_2$= | Z |

# State Estimation




Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

This is what the observer has to work with…

$N = 3$

$M = 3$

$\pi_1 = ½$ $\pi_2 = ½$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = ⅓$ $a_{13} = ⅔$

$a_{12} = ⅓$ $a_{22} = 0$ $a_{13} = ⅔$

$a_{13} = ⅓$ $a_{32} = ⅓$ $a_{13} = ⅓$

$b_1 (X) = ½$ $b_1 (Y) = ½$ $b_1 (Z) = 0$

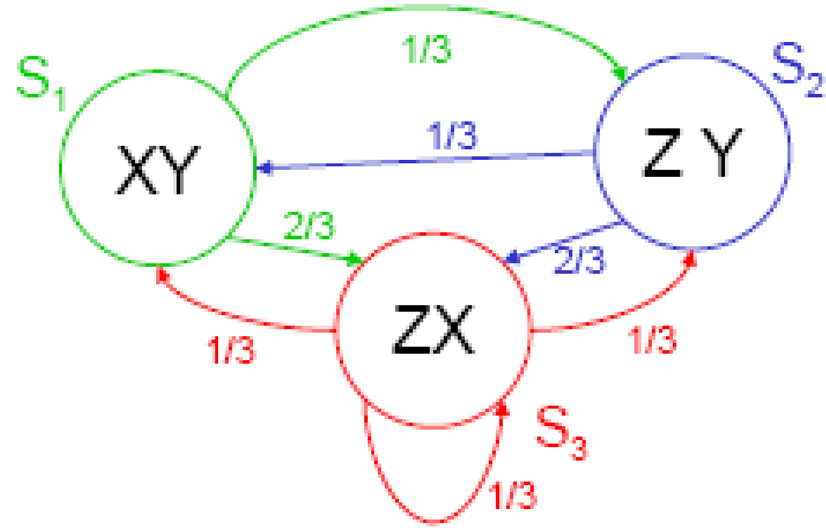
$b_2 (X) = 0$ $b_2 (Y) = ½$ $b_2 (Z) = ½$

$b_3 (X) = ½$ $b_3 (Y) = 0$ $b_3 (Z) = ½$

| | | | |
|---|---|---|---|
| $q_0$= | ? | $O_0$= | X |
| $q_1$= | ? | $O_1$= | X |
| $q_2$= | ? | $O_2$= | Z |

# Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) =$
$P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?




Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

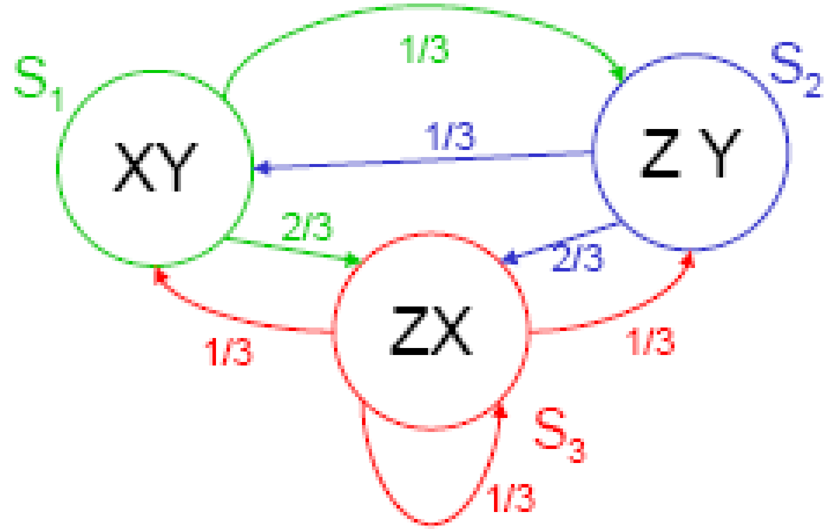
$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \mid \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

How do we compute P(O|Q) for an arbitrary path Q?

# Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\, O_2\, O_3) =$
$P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?




Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \mid \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

$P(Q) = P(q_1, q_2, q_3)$

$= P(q_1)\, P(q_2, q_3 | q_1)$ (chain rule)

$= P(q_1)\, P(q_2 | q_1)\, P(q_3 | q_2, q_1)$ (chain)

$= P(q_1)\, P(q_2 | q_1)\, P(q_3 | q_2)$ (why?)

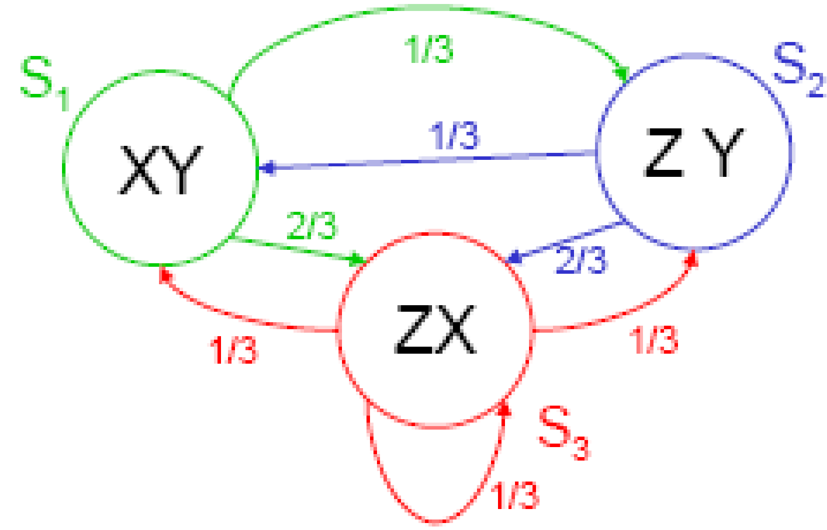
Example in the case $Q = S_1\, S_3\, S_3$:

$= 1/2 * 2/3 * 1/3 = 1/9$

How do we compute P(O|Q) for an arbitrary path Q?

# Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) =$
$P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?




Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \mid \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

How do we compute P(O|Q) for an arbitrary path Q?

P(O|Q)

$= P(O_1\ O_2\ O_3\ |q_1\ q_2\ q_3\ )$

$= P(O_1\ |\ q_1\ )\ P(O_2\ |\ q_2\ )\ P(O_3\ |\ q_3\ )$ (why?)

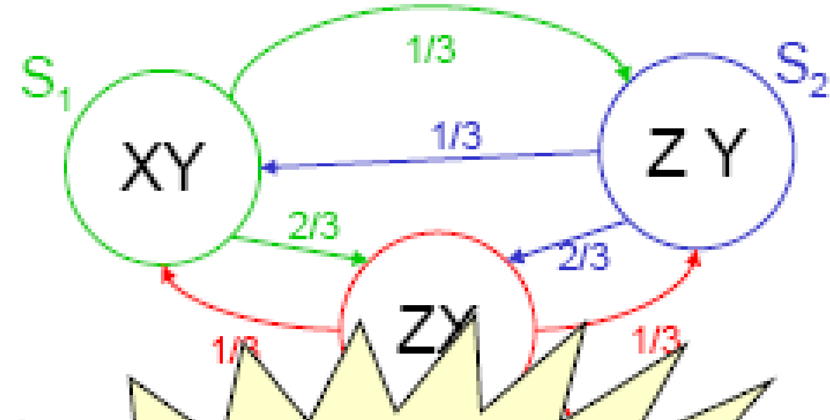
Example in the case $Q = S_1\ S_3\ S_3$:

$= P(X|\ S_1)\ P(X|\ S_3)\ P(Z|\ S_3) =$

$=1/2 * 1/2 * 1/2 = 1/8$

# Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) = P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?




Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} | \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

How do we compute P(O|Q) for an arbitrary path Q?

P(**O**) would need 27 P(Q) computations and 27 P(O|Q) computations

A sequence of 20 observations would need $3^{20}$ = 3.5 billion computations and 3.5 billion P(O|Q) computations

So let's be smarter…

# The Prob. of a given series of observations, non-exponential-cost-style

Given observations $O_1\ O_2\ \dots\ O_T$

Define

$$\alpha_t(i) = P(O_1\ O_2\ \dots\ O_t\ \wedge q_t = S_i \mid \lambda) \qquad \text{where } 1 \le t \le T$$

$\alpha_t(i)$ = Probability that, in a random trial,

- We'd have seen the first t observations
- We'd have ended up in $S_i$ as the t'th state visited.

In our example, what is $\alpha_2(3)$ ?

# $\alpha_t(i)$: easy to define recursively

$\alpha_t(i) = P(O_1\ O_2\ \ldots\ O_T\ \wedge q_t = S_i \mid \lambda)$ ($\alpha_t(i)$ can be defined stupidly by considering all paths length "t". How?)

$$\alpha_1(i) = P(O_1 \wedge q_1 = S_i)$$
$$= P(q_1 = S_i) P(O_1 | q_1 = S_i)$$
$$= \qquad \text{what?}$$
$$\alpha_{t+1}(j) = P(O_1 O_2 \ldots O_t O_{t+1} \wedge q_{t+1} = S_j)$$
$$=$$

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$$= \qquad \text{what?}$$

$$\alpha_{t+1}(j) = P(O_1 O_2 \ldots O_t O_{t+1} \wedge q_{t+1} = S_j)$$

$$= \sum_{i=1}^{N} P(O_1 O_2 \ldots O_t \wedge q_t = S_i \wedge O_{t+1} \wedge q_{t+1} = S_j)$$

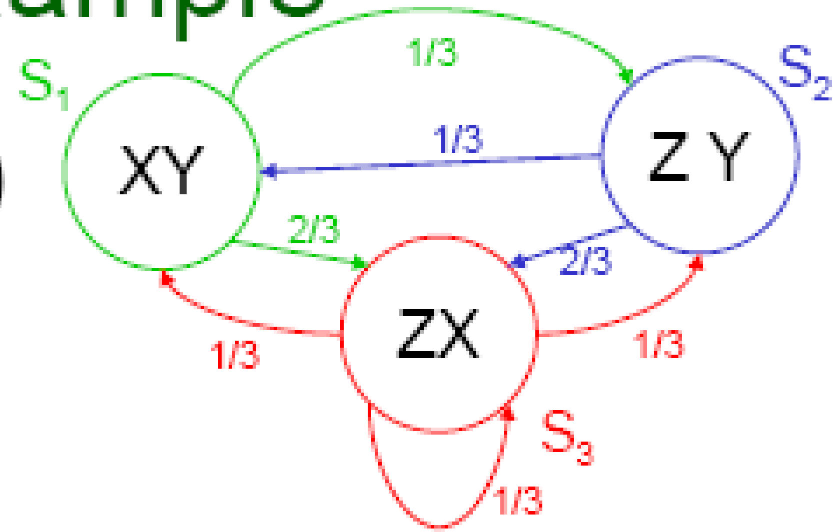
$$= \sum_{i=1}^{N} P(O_{t+1}, q_{t+1} = S_j | O_1 O_2 \ldots O_t \wedge q_t = S_i) P(O_1 O_2 \ldots O_t \wedge q_t = S_i)$$

$$= \sum_{i} P(O_{t+1}, q_{t+1} = S_j | q_t = S_i) \alpha_t(i)$$

$$= \sum_{i} P(q_{t+1} = S_j | q_t = S_i) P(O_{t+1} | q_{t+1} = S_j) \alpha_t(i)$$

$$= \sum_{i} a_{ij} b_j(O_{t+1}) \alpha_t(i)$$

# in our example




$$\alpha_t(i) = P(O_1 O_2 .. O_t \wedge q_t = S_i | \lambda)$$

$$\alpha_1(i) = b_i(O_1)\pi_i$$

$$\alpha_{t+1}(j) = \sum_i a_{ij} b_j(O_{t+1}) \alpha_t(i)$$

WE SAW $O_1\, O_2\, O_3$ = X X Z

$$\alpha_1(1) = \frac{1}{4} \qquad \alpha_1(2) = 0 \qquad \alpha_1(3) = 0$$

$$\alpha_2(1) = 0 \qquad \alpha_2(2) = 0 \qquad \alpha_2(3) = \frac{1}{12}$$

$$\alpha_3(1) = 0 \qquad \alpha_3(2) = \frac{1}{72} \qquad \alpha_3(3) = \frac{1}{72}$$

# Easy Question

We can cheaply compute

$$\alpha_t(i)=P(O_1O_2\ldots O_t \wedge q_t=S_i)$$

(How) can we cheaply compute

$$P(O_1O_2\ldots O_t) \quad ?$$

(How) can we cheaply compute

$$P(q_t=S_i|O_1O_2\ldots O_t)$$

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(How) can we cheaply compute

$$P(O_1O_2\ldots O_t) \quad ?$$

$$\sum_{i=1}^{N} \alpha_t(i)$$

(How) can we cheaply compute

$$P(q_t=S_i|O_1O_2\ldots O_t)$$

$$\frac{\alpha_t(i)}{\sum_{j=1}^{N} \alpha_t(j)}$$

# Most probable path given observations

What's most probable path given $O_1 O_2 \ldots O_T$, i.e.

What is $\underset{Q}{\operatorname{argmax}} \mathrm{P}(Q \mid O_1 O_2 \ldots O_T)$?

Slow, stupid answer :

$$\underset{Q}{\operatorname{argmax}} \mathrm{P}(Q \mid O_1 O_2 \ldots O_T)$$

$$= \underset{Q}{\operatorname{argmax}} \frac{\mathrm{P}(O_1 O_2 \ldots O_T \mid Q)\mathrm{P}(Q)}{\mathrm{P}(O_1 O_2 \ldots O_T)}$$

$$= \underset{Q}{\operatorname{argmax}} \mathrm{P}(O_1 O_2 \ldots O_T \mid Q)\mathrm{P}(Q)$$

# Efficient MPP computation

We're going to compute the following variables:

$$\delta_t(i) = \max_{q_1 q_2 .. q_{t-1}} P(q_1\ q_2\ ..\ q_{t-1} \wedge q_t = S_i \wedge O_1\ ..\ O_t)$$

= The Probability of the path of Length t-1 with the maximum chance of doing all these things:

…OCCURING

and

…ENDING UP IN STATE $S_i$

and

…PRODUCING OUTPUT $O_1 \ldots O_t$

DEFINE: $mpp_t(i)$ = that path

So: $\delta_t(i) = \text{Prob}(mpp_t(i))$

# The Viterbi Algorithm

$$\delta_t(i) = \max_{q_1 q_2 \ldots q_{t-1}} P(q_1 q_2 \ldots q_{t-1} \wedge q_t = S_i \wedge O_1 O_2 .. O_t)$$

$$mpp_t(i) = \arg\max_{q_1 q_2 \ldots q_{t-1}} P(q_1 q_2 \ldots q_{t-1} \wedge q_t = S_i \wedge O_1 O_2 .. O_t)$$

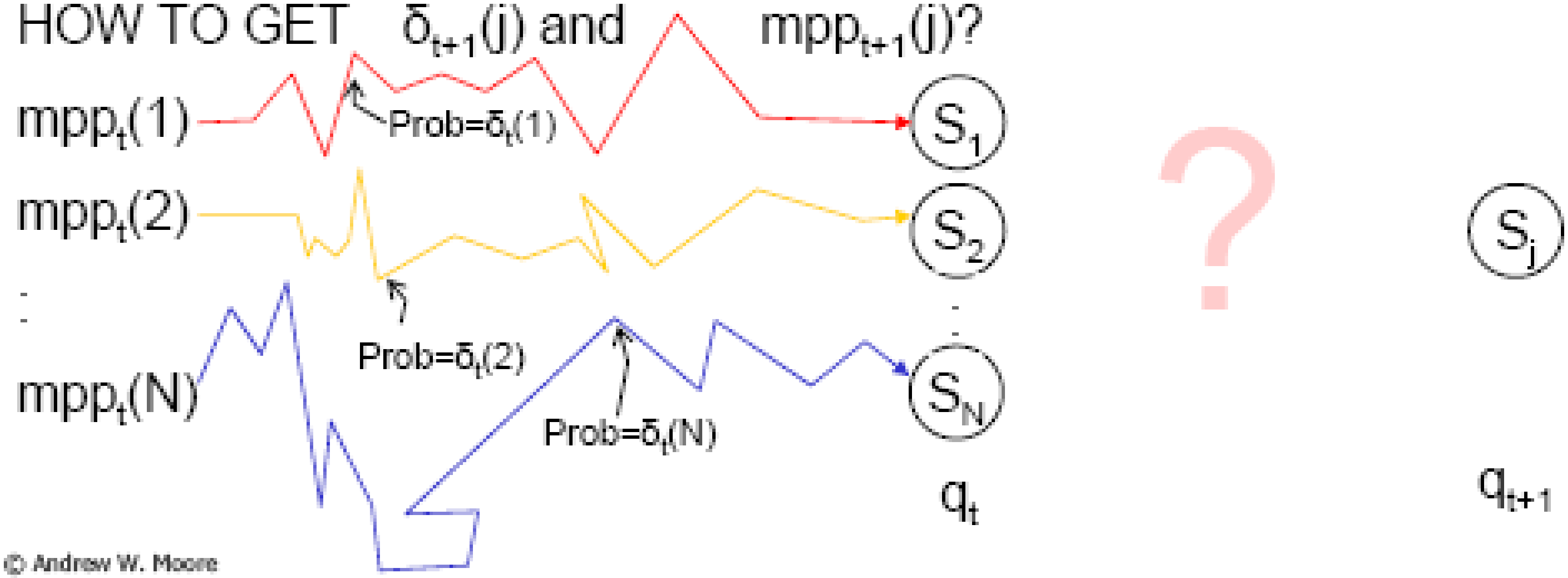
$$\delta_1(i) = \max_{\text{one choice}} P(q_1 = S_i \wedge O_1)$$

$$= P(q_1 = S_i) P(O_1 | q_1 = S_i)$$

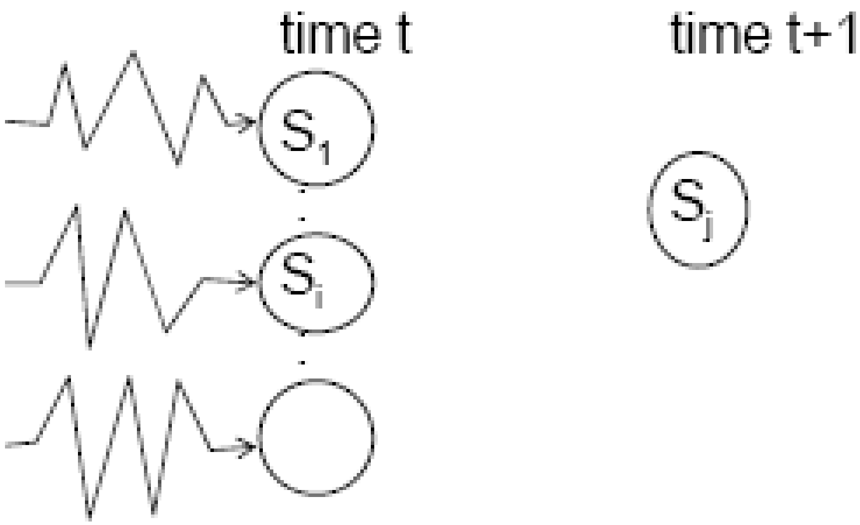
$$= \pi_i b_i(O_1)$$

Now, suppose we have all the $\delta_t(i)$'s and $mpp_t(i)$'s for all i.

HOW TO GET $\delta_{t+1}(j)$ and $mpp_{t+1}(j)$?



# The Viterbi Algorithm

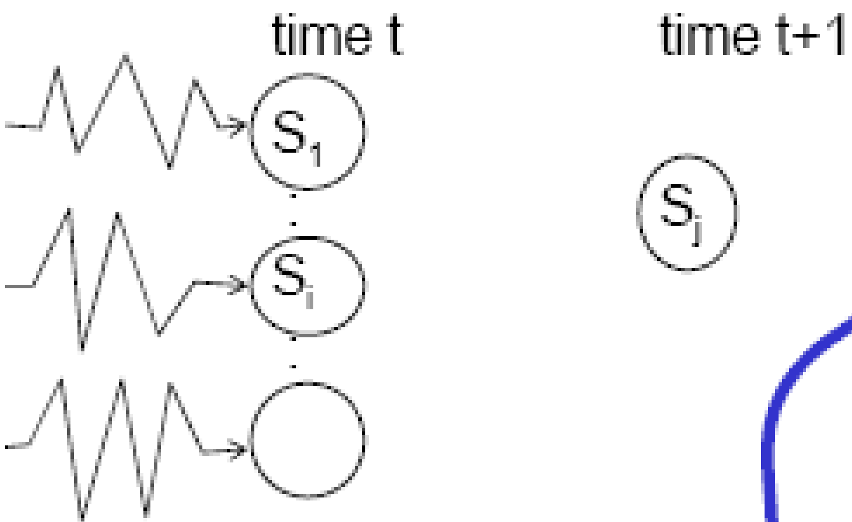



The most prob path with last two states $S_i$ $S_j$

is

the most prob path to $S_i$ , followed by transition $S_i \rightarrow S_j$

# The Viterbi Algorithm




The most prob path with last two states $S_i$ $S_j$

is

the most prob path to $S_i$ , followed by transition $S_i \rightarrow S_j$

What is the prob of that path?

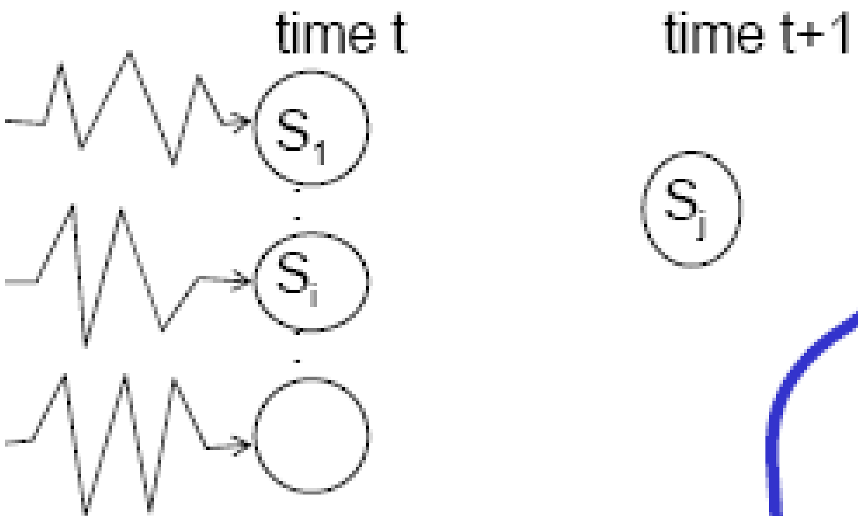
$$\delta_t(i) \times P(S_i \rightarrow S_j \wedge O_{t+1} \mid \lambda)$$

$$= \delta_t(i)\, a_{ij}\, b_j\,(O_{t+1})$$

SO The most probable path to $S_j$ has $S_{i^*}$ as its penultimate state

where $i^* = \underset{i}{\operatorname{argmax}}\ \delta_t(i)\, a_{ij}\, b_j\,(O_{t+1})$

# The Viterbi Algorithm

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SO The most probable

$S_{i^*}$ as its penultimate state

where $i^* = \underset{i}{\operatorname{argmax}}\ \delta_t(i)\, a_{ij}\, b_j\, (O_{t+1})$

Summary:

$\delta_{t+1}(j) = \delta_t(i^*)\, a_{ij}\, b_j\, (O_{t+1})$

$mpp_{t+1}(j) = mpp_{t+1}(i^*) S_{i^*}$

with i* defined to the left

# Inferring an HMM

Remember, we've been doing things like

$$P(O_1 \, O_2 \, .. \, O_T \mid \lambda)$$

That "λ" is the notation for our HMM parameters.

Now We have some observations and we want to estimate λ from them.

AS USUAL: We could use

(i) MAX LIKELIHOOD $\lambda = \underset{\lambda}{\text{argmax}} \, P(O_1 \, .. \, O_T \mid \lambda)$

(ii) BAYES

Work out $P(\lambda \mid O_1 \, .. \, O_T)$

and then take $E[\lambda]$ or $\max_{\lambda} P(\lambda \mid O_1 \, .. \, O_T)$

# Max likelihood HMM estimation

Define

$\gamma_t(i) = P(q_t = S_i \mid O_1 O_2 \ldots O_T , \lambda )$

$\varepsilon_t(i,j) = P(q_t = S_i \wedge q_{t+1} = S_j \mid O_1 O_2 \ldots O_T , \lambda )$

$\gamma_t(i)$ and $\varepsilon_t(i,j)$ can be computed efficiently $\forall i,j,t$

(Details in Rabiner paper)

$$\sum_{t=1}^{T-1} \gamma_t(i) =$$ Expected number of transitions out of state i during the path

$$\sum_{t=1}^{T-1} \varepsilon_t(i,j) =$$ Expected number of transitions from state i to state j during the path

# HMM estimation

$$\gamma_t(i) = P(q_t = S_i | O_1 O_2 .. O_T, \lambda)$$

$$\varepsilon_t(i,j) = P(q_t = S_i \wedge q_{t+1} = S_j | O_1 O_2 .. O_T, \lambda)$$

$$\sum_{t=1}^{T-1} \gamma_t(i) = \text{expected number of transitions out of state i during path}$$

$$\sum_{t=1}^{T-1} \varepsilon_t(i,j) = \text{expected number of transitions out of i and into j during path}$$

$$\text{Notice } \frac{\sum_{t=1}^{T-1} \varepsilon_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)} = \frac{\begin{pmatrix}\text{expected frequency} \\ i \rightarrow j\end{pmatrix}}{\begin{pmatrix}\text{expected frequency} \\ i\end{pmatrix}}$$

$$= \text{Estimate of Prob}(\text{Next state } S_j | \text{This state } S_i)$$

We can re-estimate

$$a_{ij} \leftarrow \frac{\sum \varepsilon_t(i,j)}{\sum \gamma_t(i)}$$

We can also re-estimate

$$b_j(O_k) \leftarrow \cdots \qquad \text{(See Rabiner)}$$

We want $a_{ij}^{\text{new}}$ = new estimate of $P(q_{t+1} = s_j \mid q_t = s_i)$

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$$= \frac{\text{Expected \# transitions } i \rightarrow j \mid \lambda^{old}, O_1, O_2, \cdots O_T}{\sum_{k=1}^{N} \text{Expected \# transitions } i \rightarrow k \mid \lambda^{old}, O_1, O_2, \cdots O_T}$$

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$$= \frac{\sum_{t=1}^{T} P(q_{t+1} = s_j, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}{\sum_{k=1}^{N} \sum_{t=1}^{T} P(q_{t+1} = s_k, q_t = s_i \mid \lambda^{\text{old}}, O_1, O_2, \cdots O_T)}$$

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$$= \frac{S_{ij}}{\sum_{k=1}^{N} S_{ik}} \text{ where } S_{ij} = \sum_{t=1}^{T} P(q_{t+1}=s_j, q_t=s_i, O_1, \cdots O_T \mid \lambda^{\text{old}})$$

$$= \text{What?}$$

We want $a_{ij}^{\text{new}}$ = new estimate of $P(q_{t+1} = s_j \mid q_t = s_i)$

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$$= a_{ij} \sum_{t=1}^{T} \alpha_t(i) \beta_{t+1}(j) b_j(O_{t+1})$$

We want $a_{ij}^{\text{new}} = S_{ij} \Big/ \sum_{k=1}^{N} S_{ik}$ where $S_{ij} = a_{ij} \sum_{t=1}^{T} \alpha_t(i) \beta_{t+1}(j) b_j(O_{t+1})$

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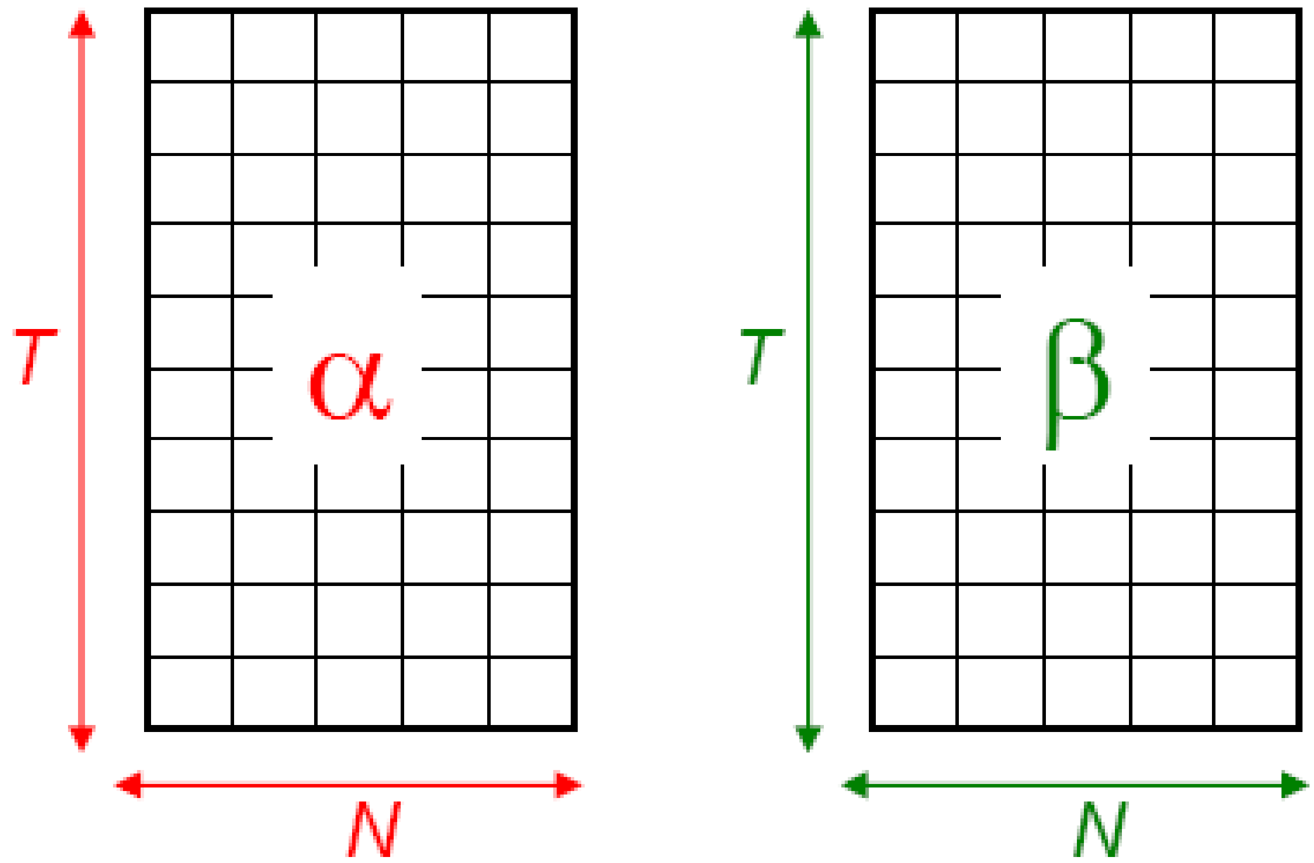


# EM for HMMs

If we knew $\lambda$ we could estimate EXPECTATIONS of quantities such as

Expected number of times in state i

Expected number of transitions $i \rightarrow j$

If we knew the quantities such as

Expected number of times in state i

Expected number of transitions $i \rightarrow j$

We could compute the MAX LIKELIHOOD estimate of

$$\lambda = \langle \{a_{ij}\}, \{b_i(j)\}, \pi_i \rangle$$

Roll on the EM Algorithm…

# EM 4 HMMs

1. Get your observations $O_1 \ldots O_T$
2. Guess your first $\lambda$ estimate $\lambda(0)$, k=0
3. k = k+1
4. Given $O_1 \ldots O_T$, $\lambda(k)$ compute
   $\gamma_t(i)$ , $\varepsilon_t(i,j)$ $\forall 1 \le t \le T$, $\forall 1 \le i \le N$, $\forall 1 \le j \le N$
5. Compute expected freq. of state i, and expected freq. $i \rightarrow j$
6. Compute new estimates of $a_{ij}$, $b_j(k)$, $\pi_i$ accordingly. Call them $\lambda(k+1)$
7. Goto 3, unless converged.

- **Also known (for the HMM case) as the BAUM-WELCH algorithm.**

# Bad News

- There are lots of local minima

# Good News

- The local minima are usually adequate models of the data.

# Notice

- EM does not estimate the number of states. That must be given.
- Often, HMMs are forced to have some links with zero probability. This is done by setting $a_{ij}=0$ in initial estimate $\lambda(0)$
- Easy extension of everything seen today: HMMs with real valued outputs

# Bad News

- There are lots o
- The local minim
  data.

Trade-off between too few states (inadequately modeling the structure in the data) and too many (fitting the noise).

Thus #states is a regularization parameter.

Blah blah blah… bias variance tradeoff…blah blah…cross-validation…blah blah….AIC, BIC….blah blah (same ol' same ol')

# Notice

- EM does not estimate the number of states. That must be given.
- Often, HMMs are forced to have some links with zero probability. This is done by setting $a_{ij}=0$ in initial estimate $\lambda(0)$
- Easy extension of everything seen today: HMMs with real valued outputs

# What You Should Know

- What is an HMM ?
- Computing (and defining) $\alpha_t(i)$
- The Viterbi algorithm
- Outline of the EM algorithm
- To be very happy with the kind of maths and analysis needed for HMMs
- Fairly thorough reading of Rabiner* up to page 266* [Up to but not including "IV. Types of HMMs"].

DON'T PANIC: starts on p. 257.

*L. R. Rabiner, "A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition," Proc. of the IEEE, Vol.77, No.2, pp.257--286, 1989.

http://ieeexplore.ieee.org/iel5/5/698/00018626.pdf?arnumber=18626

# Outline

**Introduction to Temporal Processes**

- Markov chains
- Hidden Markov models

**Discrete-State HMMs**

- Inference: Filtering, smoothing, Viterbi, classification
- Learning: EM algorithm

**Continuous-State HMMs**

- Linear state space models: Kalman filters
- Nonlinear dynamical systems: Particle filters

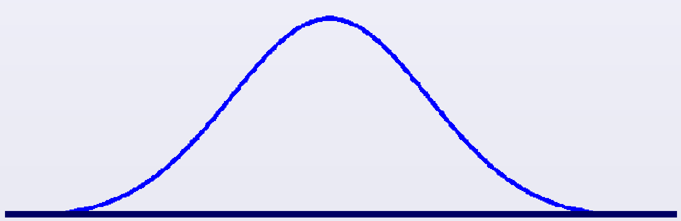
**Applications and Extensions**

# Discrete HMMs: Observations

**Discrete Observations** $y_t \in \{1, 2, \ldots, M\}$

$$p(y_t \mid x_t = 1) = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.5 \\ 0.1 \end{bmatrix} \qquad p(y_t \mid x_t = 2) = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.1 \\ 0.5 \end{bmatrix}$$

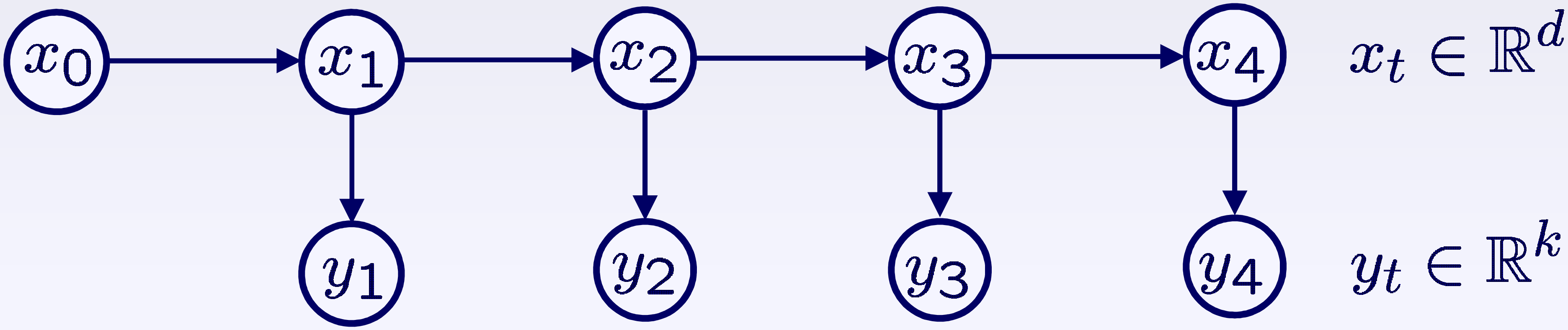
**Continuous Observations** $y_t \in \mathbb{R}^k$



$p(y_t \mid x_t = 1)$

$p(y_t \mid x_t = 2)$

# Linear State Space Models

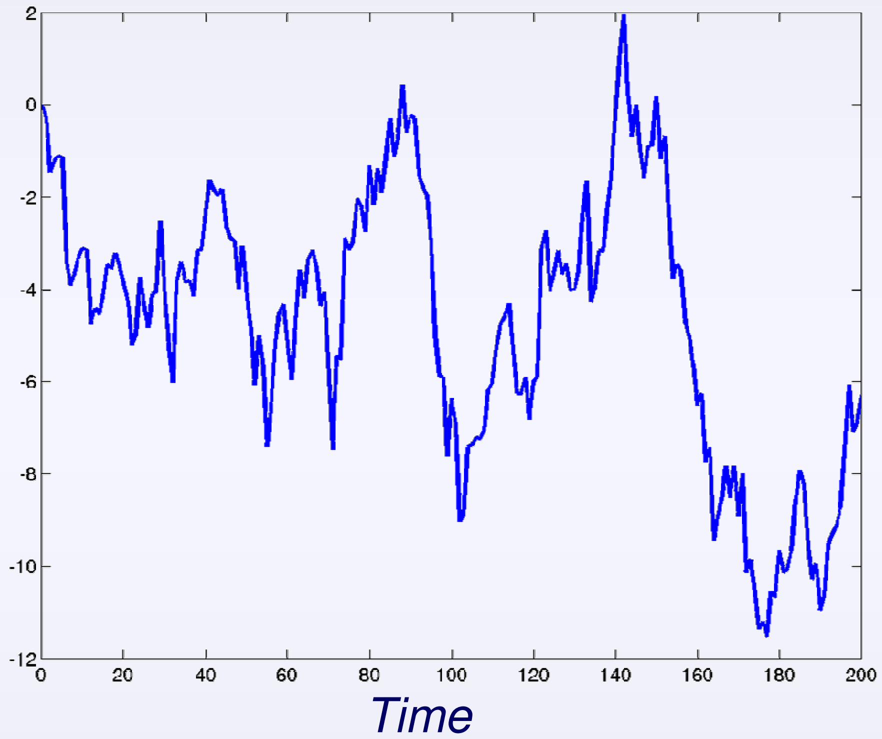



$$x_{t+1} = Ax_t + w_t \qquad w_t \sim \mathcal{N}(0, Q)$$
$$y_t = Cx_t + v_t \qquad v_t \sim \mathcal{N}(0, R)$$

- States & observations jointly Gaussian:
  - All marginals & conditionals Gaussian
  - Linear transformations remain Gaussian

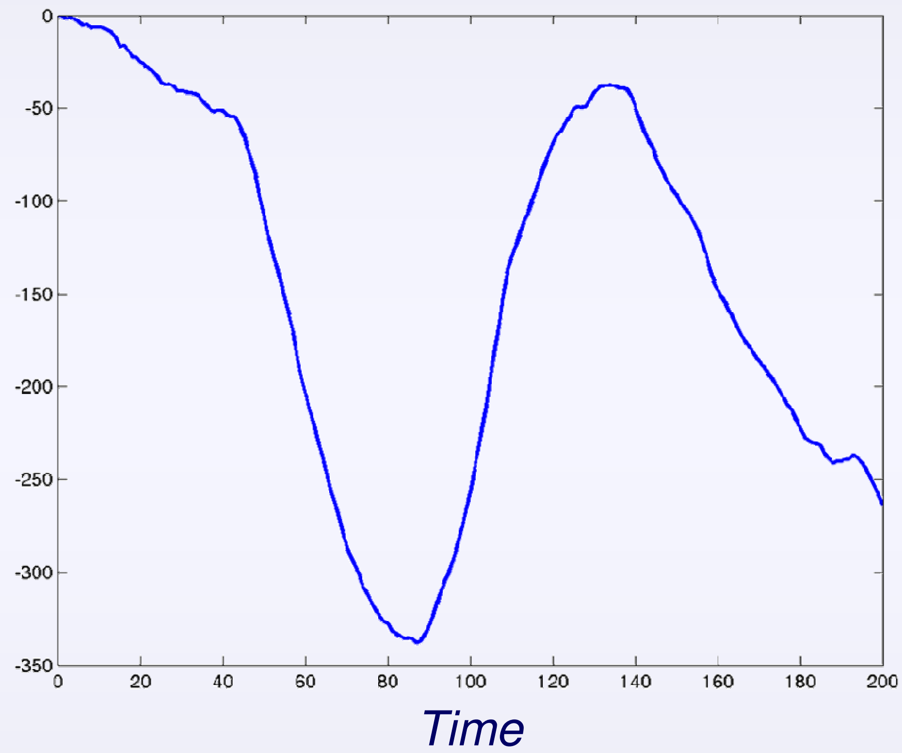
# Simple Linear Dynamics

*Brownian Motion*




$$x_{t+1} = x_t + w_t$$

*Constant Velocity*




$$\begin{bmatrix} x_{t+1} \\ \delta_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \delta_t \end{bmatrix} + w_t$$