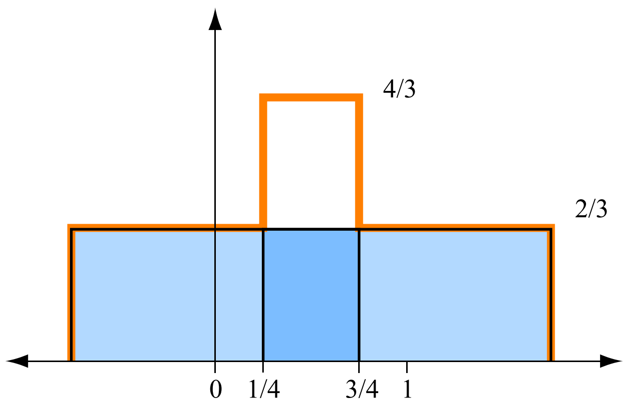
CS 465 Homework 3 Solution

Problem 1: Ripple and renormalization

1. Which of the following filters are ripple free? For those that are not, compute the convolution with c at one point where the value is not 1. The convolutions are plotted below for all points, but to get credit you only needed to evaluate the convolution at one point.

   (Note: Only a single counter example or statement that the filter is ripple free is required for a correct solution. We have included the longer answers only for clarity.)

   (a) A box of radius $\frac{3}{4}$

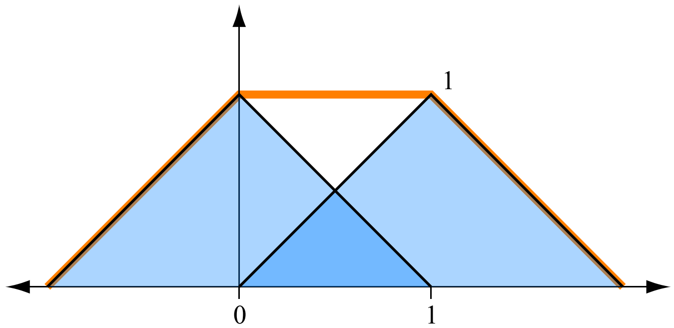
   

   The above plot is the result of $f_{\text{box},\frac{3}{4}}$ convolved with two impulses at $x = 0$ and $x = 1$. Any time you convolve a function f with an impulse sequence, you can think of it as duplicating f for every impulse and summing the duplicates into one function, the result. Between $x = \frac{1}{4}$ and $x = \frac{3}{4}$, the sum is $x = \frac{4}{3}$.

   Since in this case f has $r < 1$, you need only convolve it with two impulses to discern its behavior. If you were to convolve it with c, the result would be a square wave that looks like the above example.

   Answer: Not ripple free.

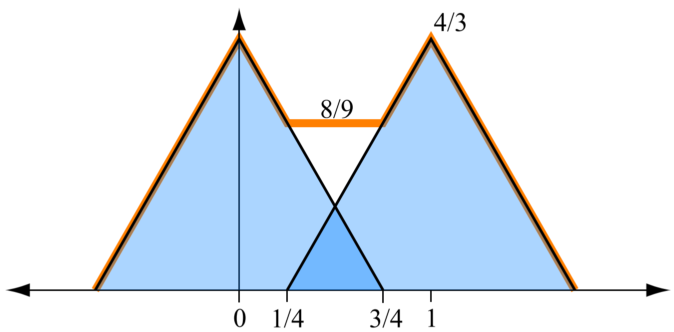
   (b) A tent of radius 1



You could answer immediately by recalling that convolution with $f_{\text{tent},1}$ is equivalent to linear interpolation, and interpolating linearly between identically valued samples produces a constant function. But here's $f_{\text{tent},1}$, convolved with two impulses again, for illustration.

Answer: Ripple free.

(c) A tent of radius $\frac{3}{4}$



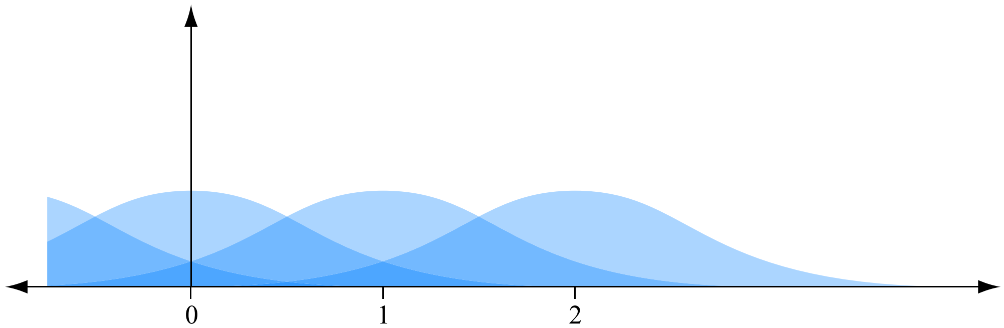
Answer: Not ripple free.

2. Prove any ripple free filter is normalized.

We have a mathematical definition of "ripple free" from the homework assignment, illustrated nicely by figure 4.27 on page 92 of the textbook:

$$\sum_i f(x-i) = 1 \quad \forall x \tag{1}$$

Notice that the infinite sum is equivalent to $(c \star f)$. The following is a plot of $f(x+1) + f(x) + f(x-1) + f(x-2)$, which is a few terms of the infinite sum in (1). On the interval $[0, 1]$, several (we don't necessarily know how many) filter instances overlap.



From (1), we have:

$$\int_0^1 \sum_i f(x-i)dx = \int_0^1 1dx = 1$$
$$= \sum_i \int_0^1 f(x-i)dx = 1$$

Establishing that this sum of integrals is 1 suggests that on every 1-unit-wide interval, there is a total of 1 in filter weight. If you look at the tent filter, for example, you can convince yourself that the original filter can be reassembled from the chunks in a 1-unit-wide interval. So in some sense, since every integer position has a filter instance centered on it, and every integer position corresponds to 1 unit of filter weight, then each filter instance must be "worth" exactly 1 in filter weight, meaning it is normalized. We can formalize this idea by doing of change of variables in the above integral (where $y = x - i$):

$$= \sum_i \int_{-i}^{-i+1} f(y)dy = 1$$
$$= \int_{-\infty}^{\infty} f(y)dy = 1$$

Beginning from the fact that f is ripple free, we have shown that f must be normalized.

Alternatively, you can directly show that a filter is normalized under the assumption that it is ripple free.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \sum_i \int_i^{i+1} f(x)dx \\
\text{Let } x = x' + i & \\
&= \sum_i \int_0^1 f(x'+i)dx' \\
&= \int_0^1 \sum_i f(x'+i)dx' \\
\text{Let } j = -i & \\
&= \int_0^1 \sum_j f(x'-j)dx' \\
&= \int_0^1 1dx' \\
&= 1
\end{aligned}$$

The two changes of variable required seem rather mysterious but arise because the ripple free condition is a sum over all integers. This gives considerable freedom in its exact expression and many similar expressions only reorder the terms in the sum. For example:

$$\sum_i f(x-i) = \sum_i f(x+i)$$

since, in both cases for any integer j, both j and $-j$ are included in the sum. The change of signs simply switches whether the positive terms are computed first or second.

3. Prove that a ripple free filter f has the property $\hat{f}(k) = 0$ for all integers k, $k \neq 0$

Since we assume f is an even function, its Fourier transform is given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)dx$$

Rather than directly compute the value of $\hat{f}(k)$, let's compute the value of $(\widehat{f \star c})(k)$. If we do find it is zero at integer k, then f must have removed all nonzero integer frequencies, which would imply $\hat{f}(k)$ is zero. Let's formalize argument:

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\sum_i f(x-i)\right)\cos(2\pi kx)dx &= \int_{-\infty}^{\infty} \cos(2\pi kx)dx = 0 \text{ (when } k \neq 0) \\
&= \sum_i \int_{-\infty}^{\infty} f(x-i)\cos(2\pi kx)dx = 0
\end{aligned}$$

$f(x-i)$ appears in the integral, but we'd really like to look at $f(x)$, so let's do another change of variable, where $y = x - i$.

$$\begin{aligned}
\sum_i \int_{-\infty}^{\infty} f(x-i)\cos(2\pi kx)dx &= \sum_i \int_{-\infty}^{\infty} f(y)\cos(2\pi k(y+i))dy = 0 \\
&= \sum_i \int_{-\infty}^{\infty} f(y)\cos(2\pi ky)dy = 0 \text{ since } cos \text{ is } 2\pi \text{ periodic} \\
&= \sum_i \hat{f}(k) = 0 \\
&= \hat{f}(k) = 0
\end{aligned}$$

Beginning from the fact that f is ripple free, we have shown that $\hat{f}(k) = 0$ for nonzero integer k.

Alternatively you could have done this similar to the previous question.

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)dx \\
&= \sum_i \int_i^{i+1} f(x)\cos(2\pi kx)dx \\
\text{Let } x = x' + i & \\
&= \sum_i \int_0^1 f(x'+i)\cos(2\pi k(x'+i))dx' \\
&= \sum_i \int_0^1 f(x'+i)\cos(2\pi kx' + 2\pi ki))dx'
\end{aligned}$$

Here since k and i are integers, the second term in the cosine is a shift of some exact number of periods. This will not change the value of the integral over one period and can be discarded.

$$\begin{aligned}
&= \sum_i \int_0^1 f(x'+i)\cos(2\pi kx'))dx' \\
&= \int_0^1 \sum_i f(x'+i)\cos(2\pi kx'))dx' \\
&= \int_0^1 1 \cdot \cos(2\pi kx'))dx' \\
&= \frac{\sin(2\pi k)}{2\pi k}
\end{aligned}$$

This latter expression is zero for all integer values of $k \neq 0$.

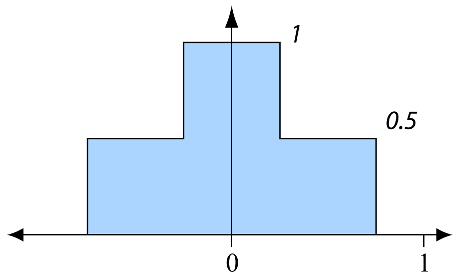
4. Write an expression for a renormalized version g of a filter f.

   Note that the denominator, the sum of the filter weights used, varies with position x.

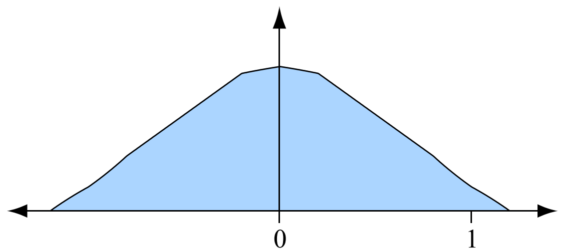
$$g(x) = \frac{f(x)}{\sum_i f(x-i)}$$

5. Plot renormalized versions of the following filters:

   (a) A box of radius $\frac{3}{4}$

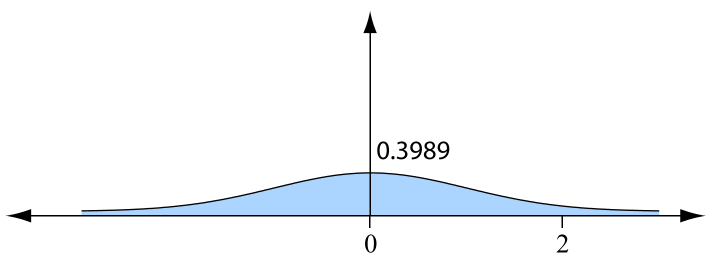


   (b) A tent of radius $\frac{6}{5}$



x	g(x)
0	0.75
1/5	0.7143
1/2	0.5
4/5	0.2857
1	0.125
6/5	0

   (c) A Gaussian with standard deviation 1



6. For a renormalized box of radius r, with $\frac{1}{2} < r < 1$, derive the Fourier transform, and plot your result for $r = \frac{3}{4}$.

   Call the renormalized box $g(x)$. Referring to the plot in question 4 of a renormalized $f_{box,\frac{3}{4}}$, you can get an idea of how the renormalized $f_{box,r}$ would look. It is essentially the sum of two $\frac{1}{2}$ height box functions, one of radius $(1 - r)$ and one of radius r. So we can compute $\hat{g}$ as the sum of the Fourier transforms of two boxes:

$$\begin{aligned}
\hat{g}(u) &= \int_{-(1-r)}^{1-r} \frac{1}{2}\cos(2\pi ux)dx + \int_{-r}^{r} \frac{1}{2}\cos(2\pi ux)dx \\
&= 2\int_{0}^{1-r} \frac{1}{2}\cos(2\pi ux)dx + 2\int_{0}^{r} \frac{1}{2}\cos(2\pi ux)dx \\
&= \int_{0}^{1-r} \cos(2\pi ux)dx + \int_{0}^{r} \cos(2\pi ux)dx \\
&= \left[\frac{\sin(2\pi ux)}{2\pi u}\right]_0^{1-r} + \left[\frac{\sin(2\pi ux)}{2\pi u}\right]_0^{r} \\
&= \frac{\sin(2\pi(1-r)u)}{2\pi u} + \frac{\sin(2\pi ru)}{2\pi u}
\end{aligned}$$

Alternatively, you could use the given expression for $\widehat{f_{box,r}}$ and the rescaling property of the Fourier transform,

$$\widehat{f(au)} = \frac{1}{|a|}\hat{f}\left(\frac{u}{a}\right)$$

to compute the Fourier transform of a sum of rescaled $f_{box,1}$ functions without doing any integrals:

$$\begin{aligned}
g(u) &= f_{box,1}\left(\frac{1}{(1-r)}x\right) + f_{box,1}\left(\frac{1}{r}x\right) \\
\hat{g}(u) &= (1-r)\widehat{f_{box,1}}\left((1-r)u\right) + r\widehat{f_{box,1}}\left(ru\right) \\
&= (1-r)\frac{\sin(2\pi(1-r)u)}{2\pi(1-r)u} + r\frac{\sin(2\pi ru)}{2\pi ru} \\
&= \frac{\sin(2\pi(1-r)u)}{2\pi u} + \frac{\sin(2\pi ru)}{2\pi u}
\end{aligned}$$