CS 465 Homework 3

(revised September 18, 2005)

out: Friday 16 September 2005
due: Friday 23 September 2005

This homework involves computing some Fourier transforms, but we will only be working with even functions (functions that are symmetric across zero, so that $f(x) = f(-x)$). For even functions, the Fourier transform can be simplified from the form given in the text (Equation 4.7) because the complex part of the exponential cancels. So for the purposes of this homework, the following equation is a definition of the Fourier transform:

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) \cos 2\pi ux \, dx \tag{1}$$

In general, for this whole homework feel free to assume that filters have finite support if it makes the reasoning and notation easier.

Problem 1: Ripple and renormalization

Recall the ripple-free property for convolution filters that are used for reconstruction: to be ripple free, a filter must produce a constant function when it is convolved with a constant sequence. If we let c be the one-dimensional constant sequence $[\ldots, 1, 1, 1, \ldots]$, the requirement for a filter f is:

$$(c \star f)(x) = 1 \quad \forall x$$

or

$$\sum_i c[i] f(x - i) = 1 \quad \forall x$$

$$\sum_i f(x - i) = 1 \quad \forall x \tag{2}$$

1. Which of the following filters are ripple free? For those that are not, compute the convolution with c at one point where the value is not 1.
   (a) A box of radius $\frac{3}{4}$
   (b) A tent of radius 1
   (c) A tent of radius $\frac{3}{4}$

2. Prove that any ripple-free filter is also a normalized filter (that is, $\int_{-\infty}^{\infty} f(x)\,dx = 1$). *Hint*: A good way to do this involves breaking the integral up into a sum of integrals over the intervals from one integer to the next.

Looking at the same situation in frequency space, the ripple-free property is simply that the filter exactly removes all nonzero multiples of the sample frequency. In our example the sample frequency is 1, so this can be written as:

$$\hat{f}(i) = 0 \quad \forall \text{ integers } i \neq 0 \tag{3}$$

3. Show that property (2) implies property (3). That is, start with the definition of the Fourier transform and (2), then compute the value of $\hat{f}$ at all integer frequencies.

Given any filter, even one that is not normalized or ripple free, we can ensure that we reconstruct constant functions properly by renormalizing the filter weights for every reconstruction computation. That is, when we evaluate the reconstructed signal at a particular point, we add up all the filter weights we used, then divide the result by that sum.

4. This renormalization process is exactly equivalent to convolution with a different reconstruction filter. Write an expression for this filter.

5. Plot the renormalized filters that result from the following filters:

   (a) A box of radius $\frac{3}{4}$

   (b) A tent of radius $\frac{6}{5}$

   (c) A Gaussian with standard deviation 1

We have said on occasion that the Fourier transform of a box filter is $\operatorname{sinc} u = (\sin \pi u)/(\pi u)$. Here is a brief derivation of that result, starting with the definitions of the Fourier transform (1) and the box filter $f_{\text{box},r}$ (page 89 in the text).

When we substitute the function $f_{\text{box},r}$ into (1), the finite support of the box has the effect of setting bounds on the integral. It can then be readily solved:

$$\begin{aligned}
\widehat{f_{\text{box},r}}(u) &= \int_{-r}^{r} \frac{1}{2r} \cos 2\pi u x \, dx \\
&= \frac{1}{2r} \left[\frac{\sin 2\pi u x}{2\pi u} \right]_{-r}^{r} \\
&= \frac{\sin 2\pi r u}{2\pi r u}
\end{aligned}$$

which reduces to $\operatorname{sinc} u$ when $r = \frac{1}{2}$.

6. For a renormalized box of radius r, with $\frac{1}{2} < r < 1$, derive the Fourier transform using a similar approach. Plot your result for $r = \frac{3}{4}$.