

2D Spline Curves

CS 465 Lecture 20

Motivation: smoothness

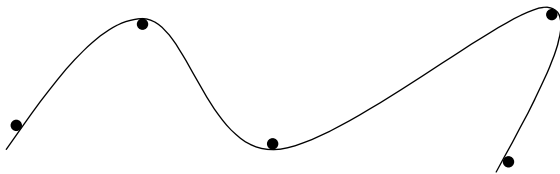
- In many applications we need smooth shapes
 - that is, without discontinuities



- So far we can make
 - things with corners (lines, squares, rectangles, ...)
 - circles and ellipses (only get you so far!)

Classical approach

- Pencil-and-paper draftsmen also needed smooth curves
- Origin of “spline:” strip of flexible metal
 - held in place by pegs or weights to constrain shape
 - traced to produce smooth contour



Translating into usable math

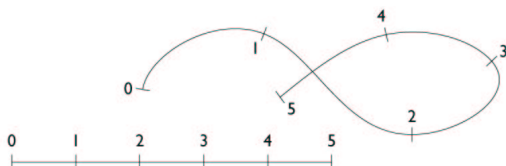
- Smoothness
 - in drafting spline, comes from physical curvature minimization
 - in CG spline, comes from choosing smooth functions
 - usually low-order polynomials
- Control
 - in drafting spline, comes from fixed pegs
 - in CG spline, comes from user-specified *control points*

Defining spline curves

- At the most general they are parametric curves

$$S = \{\mathbf{p}(t) \mid t \in [0, N]\}$$

- Generally $f(t)$ is a piecewise polynomial
 - for this lecture, the discontinuities are at the integers

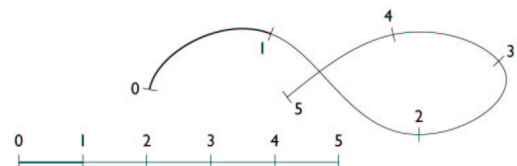


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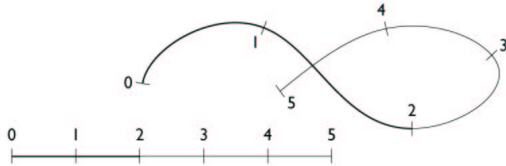


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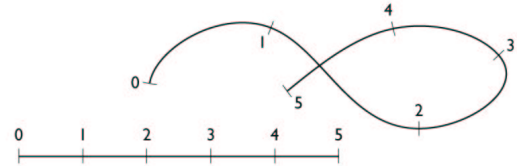
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Defining spline curves

- Generally $f(t)$ is a piecewise polynomial
 - for this lecture, the discontinuities are at the integers
 - e.g., a cubic spline has the following form over $[k, k + 1]$:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

- Coefficients are different for every interval

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Control of spline curves

- Specified by a sequence of control points
- Shape is guided by control points (aka control polygon)
 - interpolating: passes through points
 - approximating: merely guided by points



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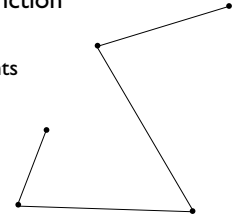


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Trivial example: piecewise linear

- This spline is just a polygon
 - control points are the vertices
- But we can derive it anyway as an illustration
- Each interval will be a linear function
 - $x(t) = at + b$
 - constraints are values at endpoints
 - $b = x_0$; $a = x_1 - x_0$
 - this is linear interpolation



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Trivial example: piecewise linear

- Vector formulation

$$x(t) = (x_1 - x_0)t + x_0$$

$$y(t) = (y_1 - y_0)t + y_0$$

$$\mathbf{p}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

- Matrix formulation

$$\mathbf{p}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

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Trivial example: piecewise linear

- Basis function formulation

- regroup expression by \mathbf{p} rather than t

$$\mathbf{p}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

$$= (1 - t)\mathbf{p}_0 + t\mathbf{p}_1$$

- interpretation in matrix viewpoint

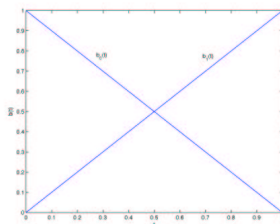
$$\mathbf{p}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

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Trivial example: piecewise linear

- Vector blending formulation: “average of points”
 - weighting functions: contribution of each point as t changes

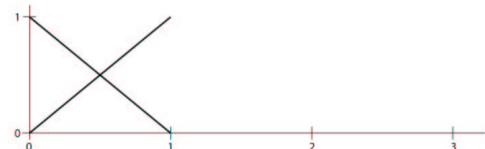


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Trivial example: piecewise linear

- Basis function formulation: “function times point”
 - basis functions: contribution of each point as t changes



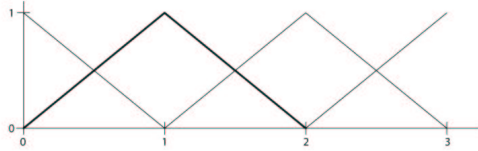
- this is just like a reconstruction filter!

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Trivial example: piecewise linear

- Basis function formulation: “function times point”
 - basis functions: contribution of each point as t changes



– this is just like a reconstruction filter!

Hermite splines

- Less trivial example
- Form of curve: piecewise cubic
- Constraints: endpoints and tangents (derivatives)



Hermite splines

- Solve constraints to find coefficients

$$\begin{aligned}
 x(t) &= at^3 + bt^2 + ct + d \\
 x'(t) &= 3at^2 + 2bt + c \\
 x(0) &= x_0 = d & d &= x_0 \\
 x(1) &= x_1 = a + b + c + d & c &= x'_0 \\
 x'(0) &= x'_0 = c & a &= 2x_0 - 2x_1 + x'_0 + x'_1 \\
 x'(1) &= x'_1 = 3a + 2b + c & b &= -3x_0 + 3x_1 - 2x'_0 - x'_1
 \end{aligned}$$

Hermite splines

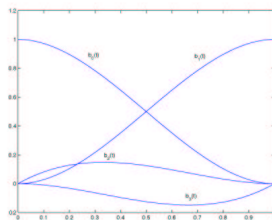
- Matrix form is much simpler

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 2 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}'_0 \\ \mathbf{p}'_1 \end{bmatrix}$$

- coefficients = rows
- basis functions = columns
 - note \mathbf{p} columns sum to $[0 \ 0 \ 0 \ 1]^T$

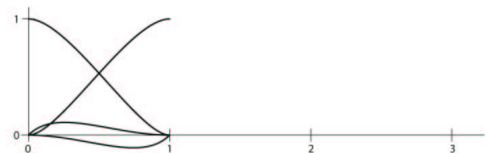
Hermite splines

- Hermite blending functions



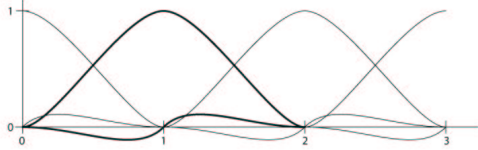
Hermite splines

- Hermite basis functions



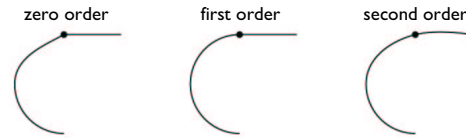
Hermite splines

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Continuity

- Smoothness can be described by degree of continuity
 - zero-order: position matches from both sides
 - first-order: tangent matches from both sides
 - second-order: curvature matches from both sides
- G_n vs. C_n



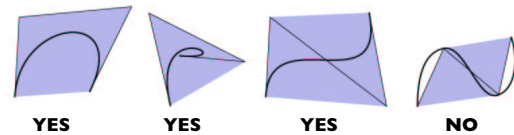
Control

- Local control
 - changing control point only affects a limited part of spline
 - without this, splines are very difficult to use
 - many likely formulations lack this
 - natural spline
 - polynomial fits



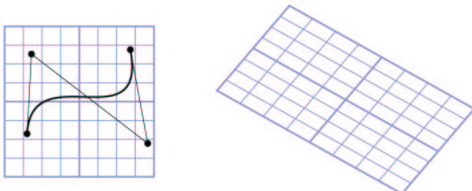
Control

- Convex hull property
 - convex hull = smallest convex region containing points
 - think of a rubber band around some pins
 - some splines stay inside convex hull of control points
 - make clipping, culling, picking, etc. simpler



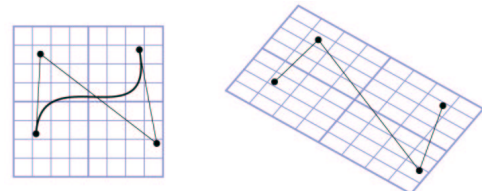
Affine invariance

- Transforming the control points is the same as transforming the curve
 - true for all commonly used splines
 - extremely convenient in practice...



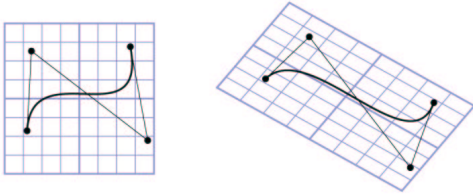
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Matrix form of spline

$$\mathbf{p}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{p}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

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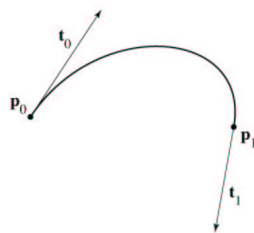
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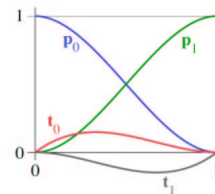
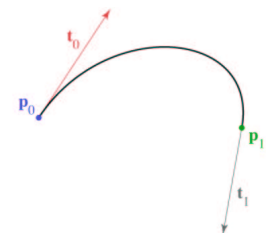
Hermite splines

- Constraints are endpoints and endpoint tangents

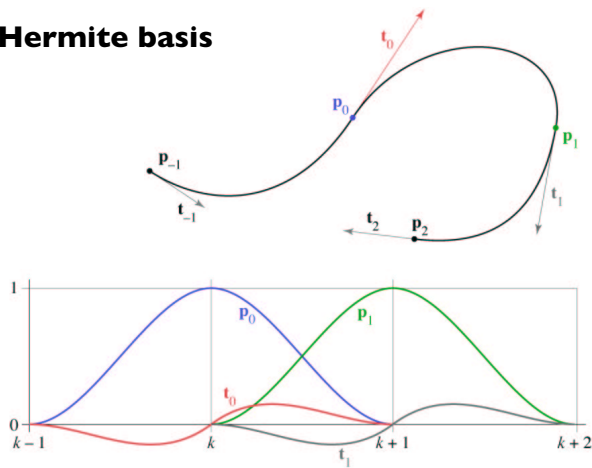


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Hermite basis

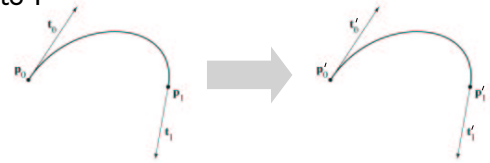


Hermite basis



Affine invariance

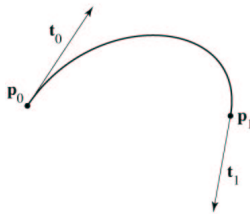
- Basis functions associated with points should always sum to 1



$$\begin{aligned} \mathbf{p}(t) &= b_0 \mathbf{p}_0 + b_1 \mathbf{p}_1 + b_2 \mathbf{v}_0 + b_3 \mathbf{v}_1 \\ \mathbf{p}'(t) &= b_0(\mathbf{p}_0 + \mathbf{u}) + b_1(\mathbf{p}_1 + \mathbf{u}) + b_2 \mathbf{v}_0 + b_3 \mathbf{v}_1 \\ &= b_0 \mathbf{p}_0 + b_1 \mathbf{p}_1 + b_2 \mathbf{v}_0 + b_3 \mathbf{v}_1 + (b_0 + b_1) \mathbf{u} \\ &= \mathbf{p}(t) + \mathbf{u} \end{aligned}$$

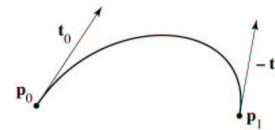
Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points



Hermite to Bézier

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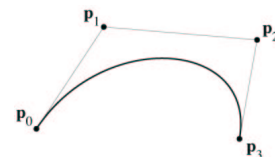
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Hermite to Bézier

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- note derivative is defined as 3 times offset
 - reason is illustrated by linear case

Hermite to Bézier

$$\begin{aligned} p_0 &= q_0 \\ p_1 &= q_3 \\ v_0 &= 3(q_1 - q_0) \\ v_1 &= 3(q_3 - q_2) \end{aligned}$$



$$\begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

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$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

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$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Bézier matrix

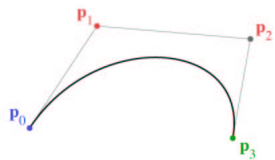
$$p(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

– note that these are the Bernstein polynomials

$$C(n,k) t^k (1-t)^{n-k}$$

and that defines Bézier curves for any degree

Bézier basis



Convex hull

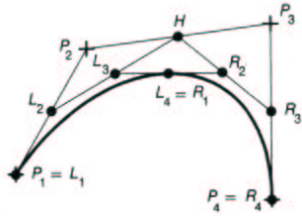
- If basis functions are all positive, the spline has the convex hull property
 - we're still requiring them to sum to 1



- if any basis function is ever negative, no convex hull prop.
 - proof: take the other three points at the same place

Subdivision

- Bézier splines can be split in two



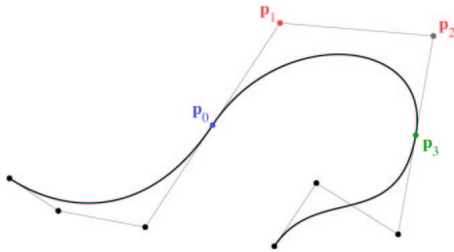
- de Casteljau's algorithm
- also works for arbitrary t

Chaining Bézier splines

- No continuity built in
- Achieve C^1 using collinear control points

Chaining Bézier splines

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- Achieve C^1 using collinear control points



Cubic Bézier splines

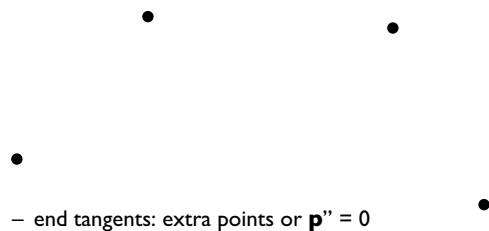
- Very widely used type, especially in 2D
 - e.g. it is a primitive in PostScript/PDF
- Can represent C^1 and/or G^1 curves with corners
- Can easily add points at any position
- Illustrator demo

Chaining spline segments

- Hermite curves are convenient because they can be made long easily
- Bézier curves are convenient because their controls are all points and they have nice properties
 - and they interpolate every 4th point, which is a little odd
- We derived Bézier from Hermite by defining tangents from control points
 - a similar construction leads to the interpolating *Catmull-Rom* spline

Hermite to Catmull-Rom

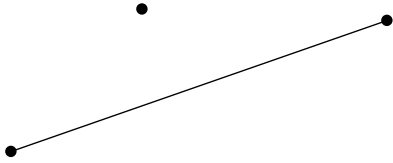
- Would like to define tangents automatically
 - use adjacent control points



- end tangents: extra points or $p'' = 0$

Hermite to Catmull-Rom

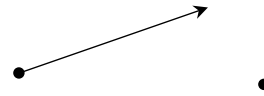
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Hermite to Catmull-Rom

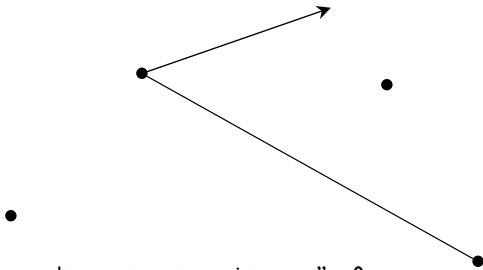
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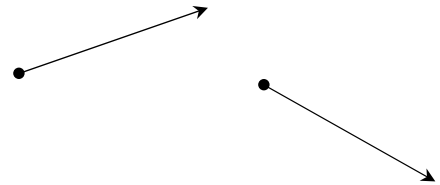
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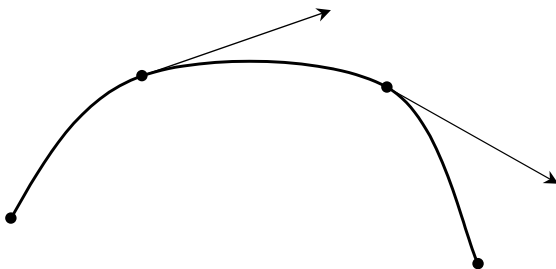
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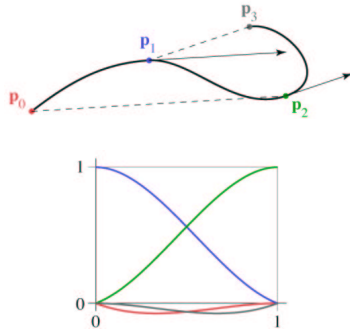
Hermite to Catmull-Rom

- Tangents are $(\mathbf{p}_{k+1} - \mathbf{p}_{k-1}) / 2$
 - scaling based on same argument about linear case

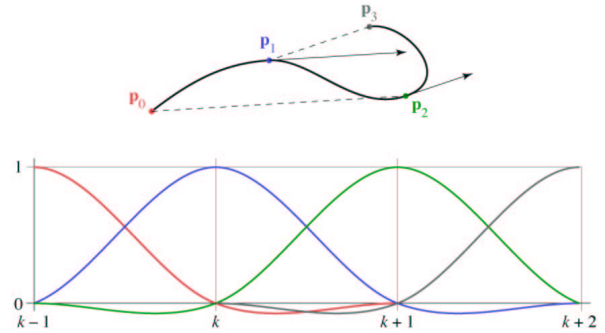
$$\begin{aligned} \mathbf{p}_0 &= \mathbf{q}_k \\ \mathbf{p}_1 &= \mathbf{q}_{k+1} \\ \mathbf{v}_0 &= 0.5(\mathbf{q}_{k+1} - \mathbf{q}_{k-1}) \\ \mathbf{v}_1 &= 0.5(\mathbf{q}_{k+2} - \mathbf{q}_k) \end{aligned}$$

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.5 & 0 & .5 & 0 \\ 0 & -0.5 & 0 & .5 \end{bmatrix} \begin{bmatrix} \mathbf{q}_{k-1} \\ \mathbf{q}_k \\ \mathbf{q}_{k+1} \\ \mathbf{q}_{k+2} \end{bmatrix}$$

Catmull-Rom basis



Catmull-Rom basis



Catmull-Rom splines

- Our first example of an interpolating spline
- Like Bézier, equivalent to Hermite
 - in fact, all splines of this form are equivalent
- First example of a spline based on just a control point sequence
- Does not have convex hull property

Converting spline representations

- All the splines we have seen so far are equivalent
 - all represented by geometry matrices
- $$\mathbf{p}_S(t) = T(t)M_S P_S$$
- where S represents the type of spline
 - therefore the control points may be transformed from one type to another using matrix multiplication

$$P_1 = M_1^{-1} M_2 P_2$$

$$\mathbf{p}_1(t) = T(t)M_1(M_1^{-1} M_2 P_2)$$

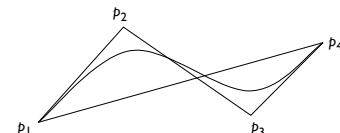
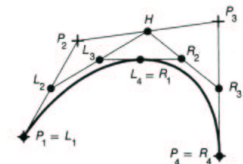
$$= T(t)M_2 P_2 = \mathbf{p}_2(t)$$

Evaluating splines for display

- Need to generate a list of line segments to draw
 - generate efficiently
 - use as few as possible
 - guarantee approximation accuracy
- Approaches
 - recursive subdivision (easy to do adaptively)
 - uniform sampling (easy to do efficiently)

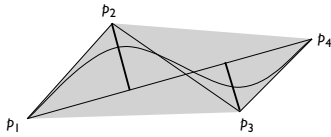
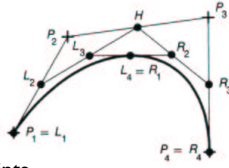
Evaluating by subdivision

- Recursively split spline
 - stop when polygon is within epsilon of curve
- Termination criteria
 - distance between control points
 - distance of control points from line



Evaluating by subdivision

- Recursively split spline
 - stop when polygon is within epsilon of curve
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Evaluating with uniform spacing

- Forward differencing
 - efficiently generate points for uniformly spaced t values
 - evaluate polynomials using repeated differences

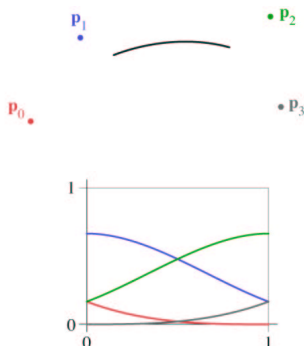
B-splines

- We may want more continuity than C^1
- We may not need an interpolating spline
- B-splines are a clean, flexible way of making long splines of arbitrary order
- Various ways to think of construction
 - a simple one is convolution
 - relationship to sampling and reconstruction

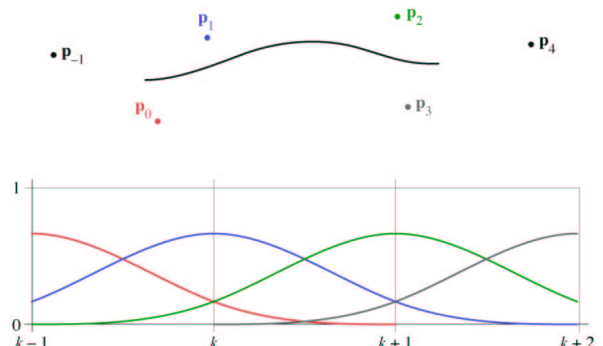
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Cubic B-spline basis



Cubic B-spline basis



Construction of B-splines

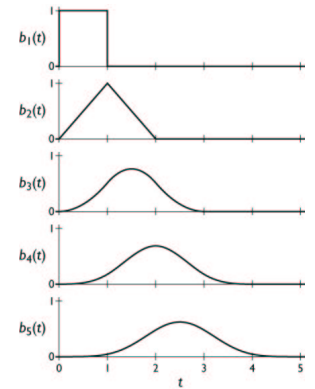
- B-splines defined for all orders
 - order d : degree $d - 1$
 - order d : d points contribute to value
- One definition: Cox-deBoor recurrence

$$b_1 = \begin{cases} 1 & 0 \leq u < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_d = \frac{t}{d-1} b_{d-1}(t) + \frac{d-t}{d-1} b_{d-1}(t-1)$$

B-spline construction

- Recurrence
 - ramp up/down
- Convolution
 - smoothing of basis fn
 - smoothing of curve



Cubic B-spline matrix

$$p(t) = [t^3 \quad t^2 \quad t \quad 1] \cdot \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{k-1} \\ \mathbf{p}_k \\ \mathbf{p}_{k+1} \\ \mathbf{p}_{k+2} \end{bmatrix}$$

Other types of B-splines

- Nonuniform B-splines
 - discontinuities not evenly spaced
 - allows control over continuity or interpolation at certain points
 - e.g. interpolate endpoints (commonly used case)
- Nonuniform Rational B-splines (NURBS)
 - ratios of nonuniform B-splines: $x(t) / w(t)$; $y(t) / w(t)$
 - key properties:
 - invariance under perspective as well as affine
 - ability to represent conic sections exactly