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Notes: Depth interpolation, perspective correct texturing

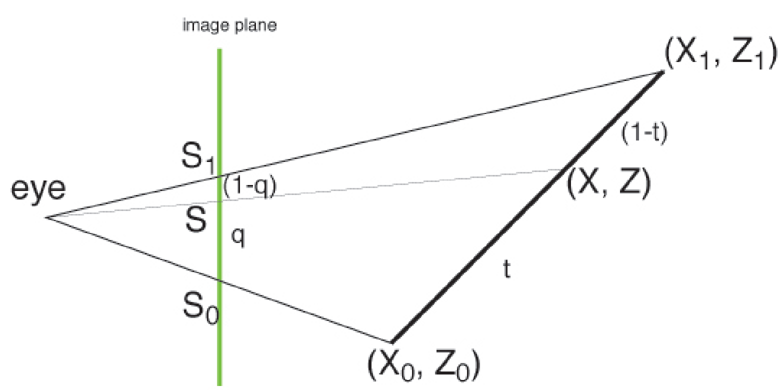
Kavita Bala

This is a short note explaining the math behind depth interpolations, and perspective correct texturing.

1 Depth interpolation and perspective-correct texturing

We want to derive the correct equations for z computation. We will stay in 2D, but 3D is an easy generalization.

Consider a 2D line with end points (X_0, Z_0) and (X_1, Z_1). These 2 points project (under a perspective transformation) to screen space points S_0 and S_1 respectively. We want to find the depth values Z', and texture coordinates U that are correct even with the perspective transform.



Now consider a point (X, Z) that lies on the line between (X_0, Z_0) and (X_1, Z_2): it's screen space projection on the image plane is S, and its texture coordinate is U. Let the screen space interpolation parameter by q, and t be the world space interpolation parameter. Our goal is to find the way we need to interpolate in screen space to produce the correct depth value and the perspective-correct U value.

Let d be the distance of the image plane from the eye. Using similar triangles for the points, we have:

$$\frac{X_0}{Z_0} = \frac{S_0}{-d} \tag{1}$$

$$\frac{X_1}{Z_1} = \frac{S_1}{-d} \tag{2}$$

Given an arbitrary point (X, Z), we have the equation for its projection S to be:

$$\frac{X}{Z} = \frac{S}{-d} \tag{3}$$

$$Z = \frac{-dX}{S} \tag{4}$$

The point S is computed by screen space interpolation of S_0 and S_1.

$$S = S_0 + q(S_1 - S_0) \tag{5}$$

X and Z are obtained by world space interpolation of X_0 and X_1, and Z_0 and Z_1, respectively.

$$[X, Z] = [X_0 + t(X_1 - X_0), Z_0 + t(Z_1 - Z_0)] \tag{6}$$

Combining Equation 4 with Equation 5 and Equation 6 we get:

$$Z = \frac{-dX}{S} \tag{7}$$

$$= \frac{-d(X_0 + t(X_1 - X_0))}{S_0 + q(S_1 - S_0)} \tag{8}$$

$$Z = \frac{-d(\frac{S_0 Z_0}{-d} + t\frac{(S_1 Z_1 - S_0 Z_0)}{-d}}{S_0 + q(S_1 - S_0)} \tag{9}$$

$$= \frac{S_0 Z_0 + t(S_1 Z_1 - S_0 Z_0)}{S_0 + q(S_1 - S_0)} \tag{10}$$

But Z is $Z_0 + t(Z_1 - Z_0)$ from Equation 4. Thus,

$$Z_0 + t(Z_1 - Z_0) = \frac{S_0 Z_0 + t(S_1 Z_1 - S_0 Z_0)}{S_0 + q(S_1 - S_0)} \tag{11}$$

which can be simplified to

$$\begin{aligned} Z_0 S_0 + Z_0 q(S_1 - S_0) + t(Z_1 - Z_0)S_0 + tq(Z_1 - Z_0)(S_1 - S_0) &= S_0 Z_0 + t(S_1 Z_1 - S_0 Z_0) \\ t[S_0 Z_1 - S_0 Z_0 + q(Z_1 - Z_0)(S_1 - S_0) - S_1 Z_1 + S_0 Z_0] &= -Z_0 q(S_1 - S_0) \\ t(S_1 - S_0)[Z_1 - q(Z_1 - Z_0)] &= Z_0 q(S_1 - S_0) \\ t[qZ_0 + (1-q)Z_1] = Z_0 q \end{aligned}$$

giving us the value of t in terms of q:

$$t = \frac{Z_0 q}{qZ_0 + (1-q)Z_1} \tag{12}$$

Substituting t in Equation 6 we get:

$$\begin{aligned} Z &= Z_0 + t(Z_1 - Z_0) = Z_0 + \frac{Z_0 q(Z_1 - Z_0)}{qZ_0 + (1-q)Z_1} && (13) \\ &= \frac{qZ_0^2 + (1-q)Z_0Z_1 + qZ_0Z_1 - qZ_0^2}{qZ_0 + (1-q)Z_1} && (14) \\ &= \frac{Z_0Z_1}{qZ_0 + (1-q)Z_1} && (15) \\ &= \frac{1}{\frac{1}{Z_0} + q(\frac{1}{Z_1} - \frac{1}{Z_0})} && (16) \end{aligned}$$

$$\begin{aligned} \frac{1}{Z} &= \frac{1}{Z_0} + q(\frac{1}{Z_1} - \frac{1}{Z_0}) && (17) \\ & && (18) \end{aligned}$$

What this equation tells us is that we can interpolate $\frac{1}{Z}$ in screen space using q to produce a value that will in fact be the correct value of $\frac{1}{Z}$. And therefore, we can use that quantity for depth comparisons in the Z-buffer to figure out occlusion.

Perspective correct texturing Our goal is to compute the perspective correct U value. Remember that in perspective projection, the Z value is moved into the W. So, $W = 1/Z$, and we get,

$$W = W_0 + q(W_1 - W_0) \quad (19)$$

This equation tells us that we can achieve that using screen-space interpolation, as long as we linearly interpolate W (which is reciprocal Z) in screen space using q, and then compute Z as $\frac{1}{W}$. For computing texture coordinates, the same substitution can be done. Except here we are interpolating the texture coordinate U. Given

$$U = U_0 + t(U_1 - U_0) \quad (20)$$

and substituting t from Equation 12 we get:

$$U = \frac{U_0W_0 + q(U_1W_1 - U_0W_0)}{W_0 + q(W_1 - W_0)} \quad (21)$$

Thus the texture coordinates U can be correctly dervived by linearly interpolating in screen space (using q) as follows: interpolate UW and W and then compute U as $\frac{interpolatedUW}{interpolatedW}$.