2D Spline Curves

CS 4620 Lecture 26

Administration

- A4 due yesterday
  - Demos? Will get back to you

- PPA2 due on Monday

- CS 4621 has a project discussion today

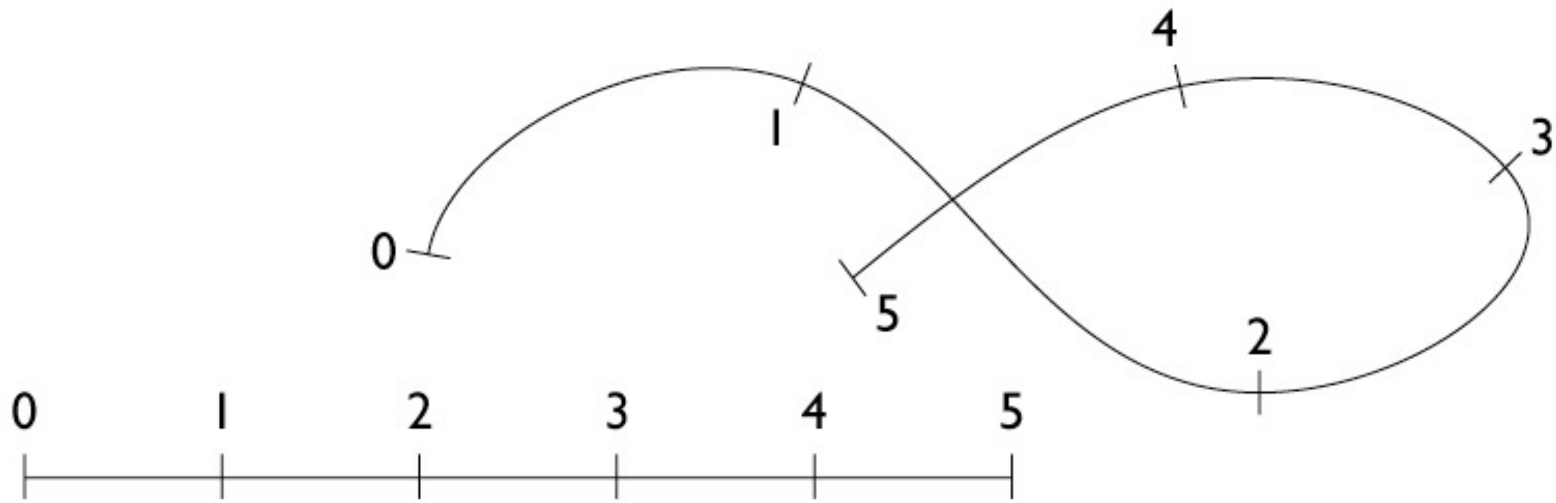
- A5 out on Monday

Defining spline curves

- At the most general they are parametric curves

$$S = \{\mathbf{f}(t) \mid t \in [0, N]\}$$

- For splines, **f**(*t*) is piecewise polynomial
  – for this lecture, the discontinuities are at the integers



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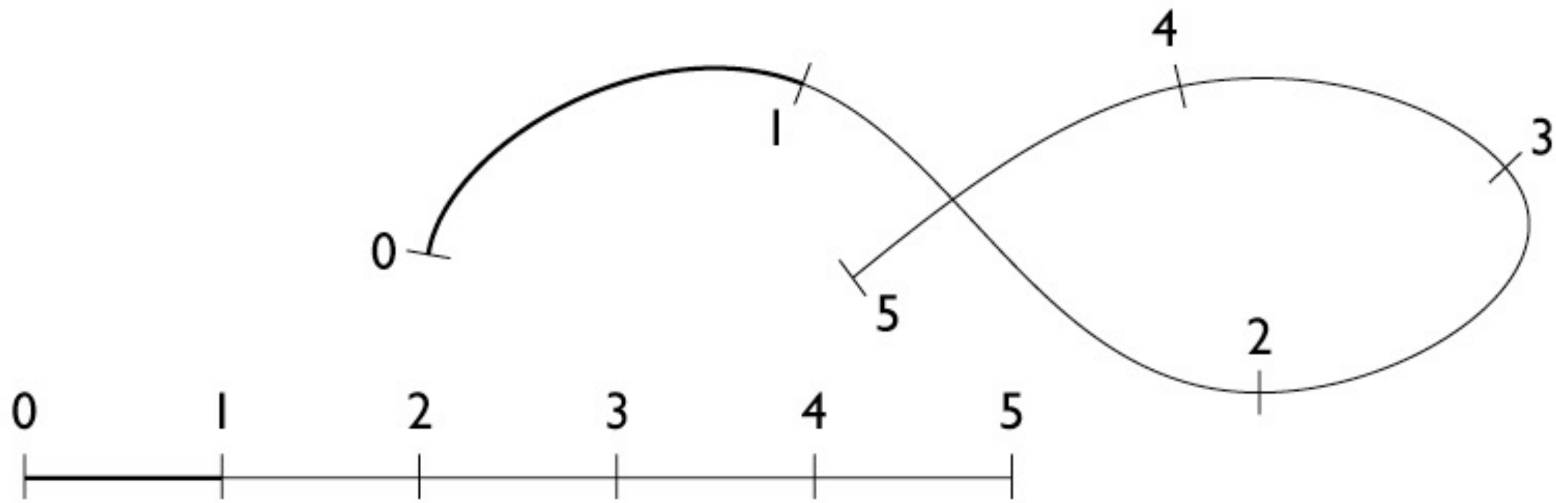

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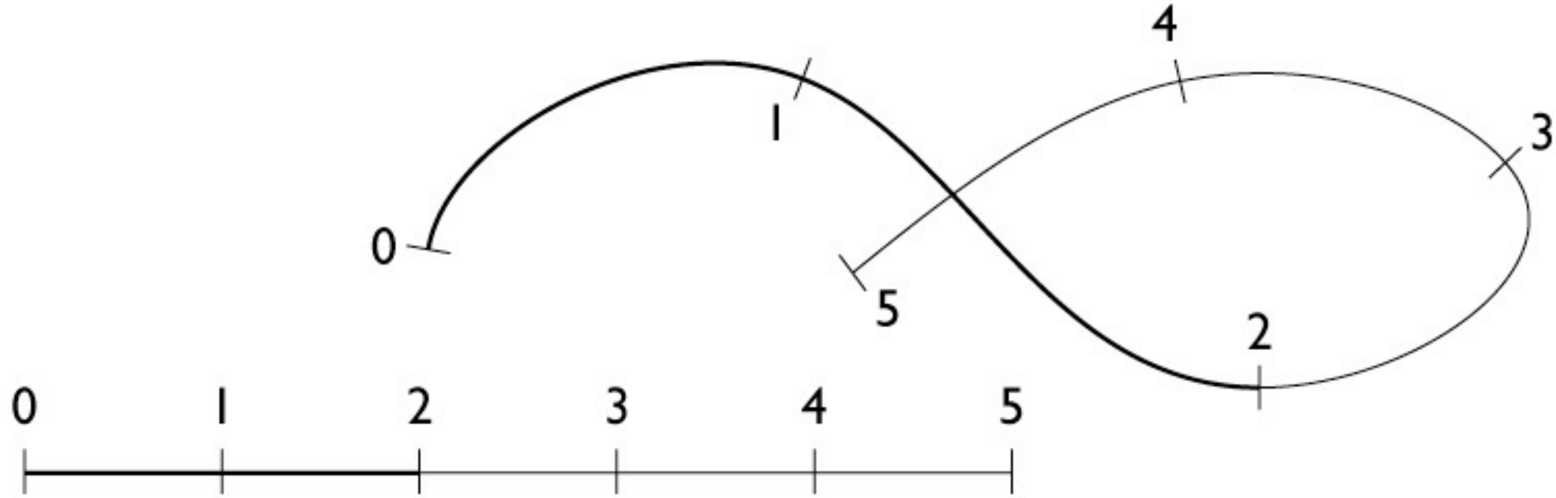

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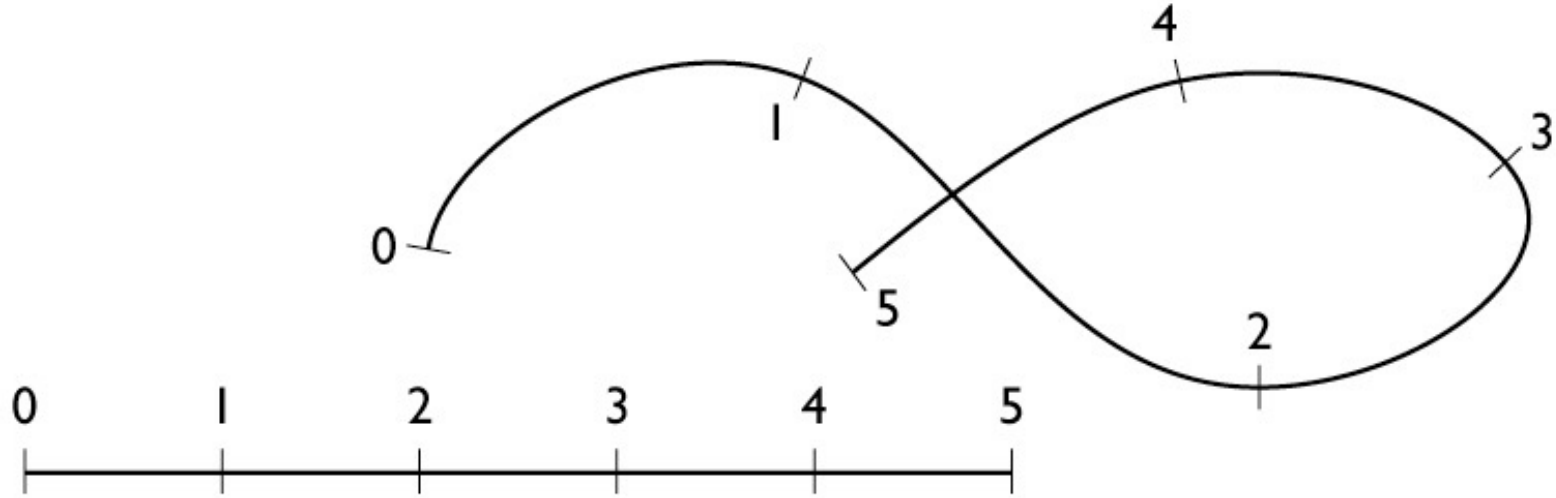


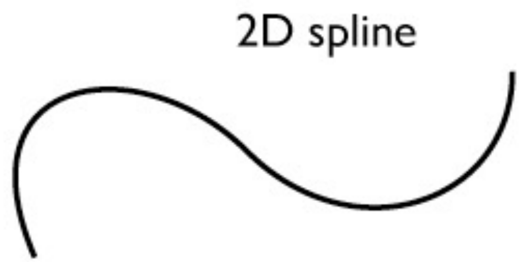
Defining spline curves

- Generally **f**(*t*) is a piecewise polynomial
  - for this lecture, the discontinuities are at the integers
  - e.g., a cubic spline has the following form over [k, k + 1]:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

  - Coefficients are different for every interval

Coordinate functions



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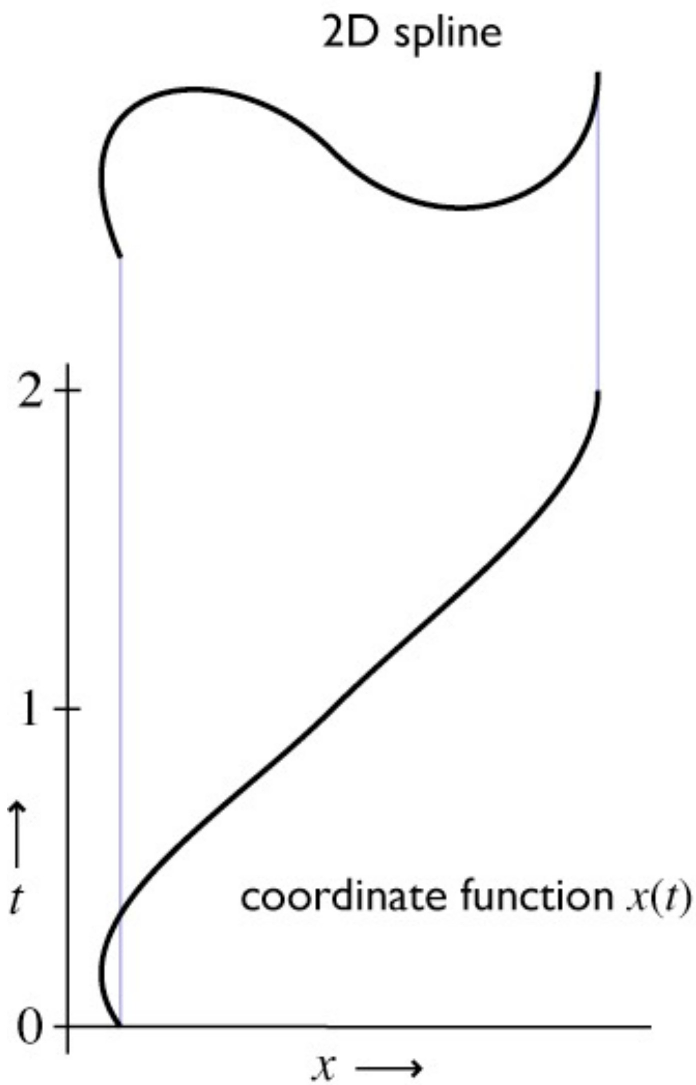

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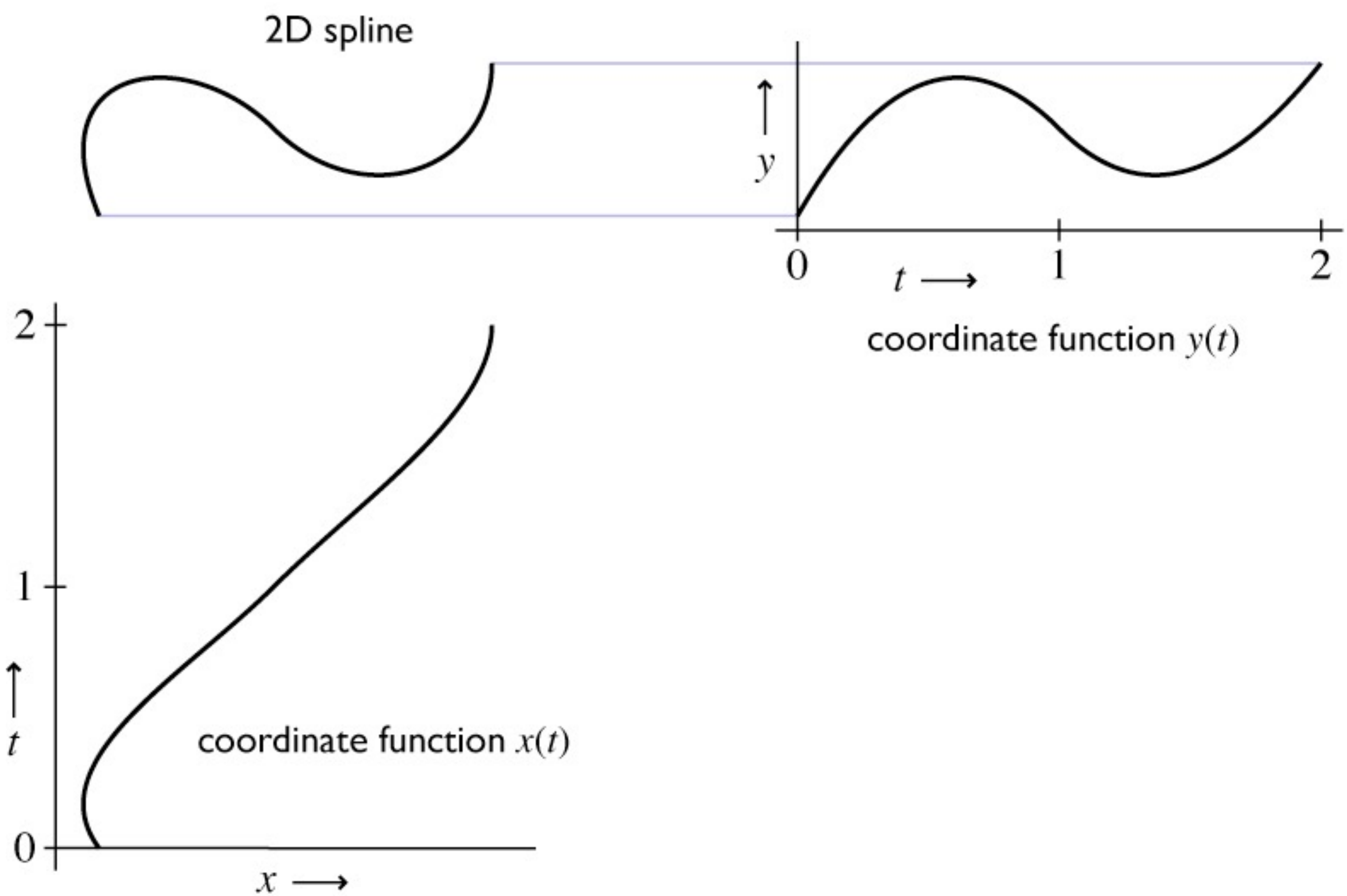

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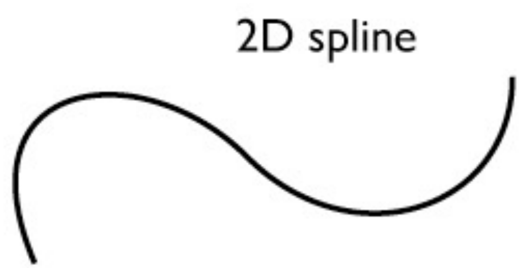

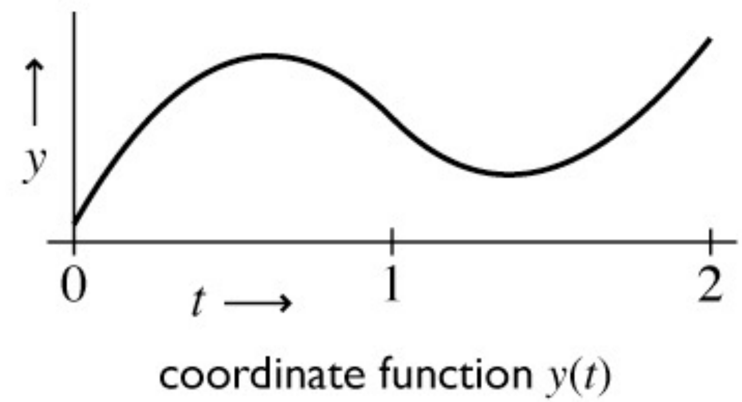

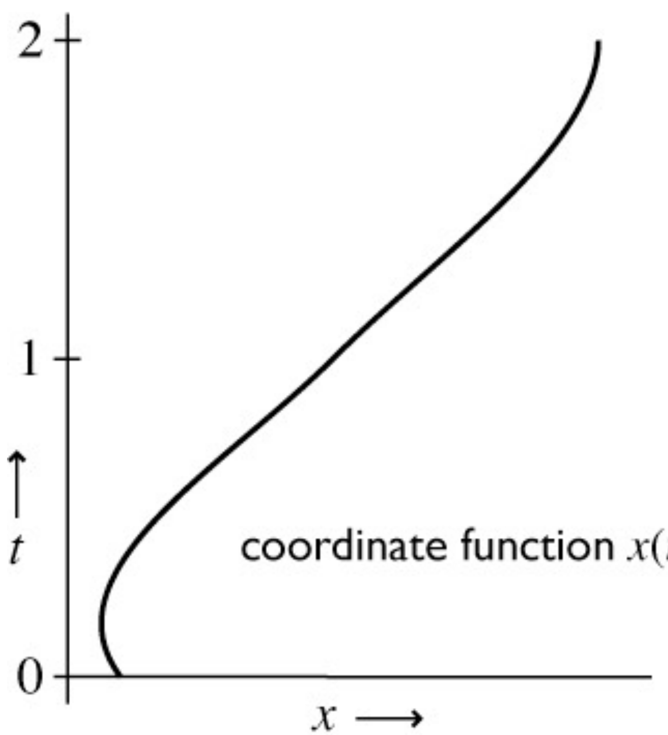

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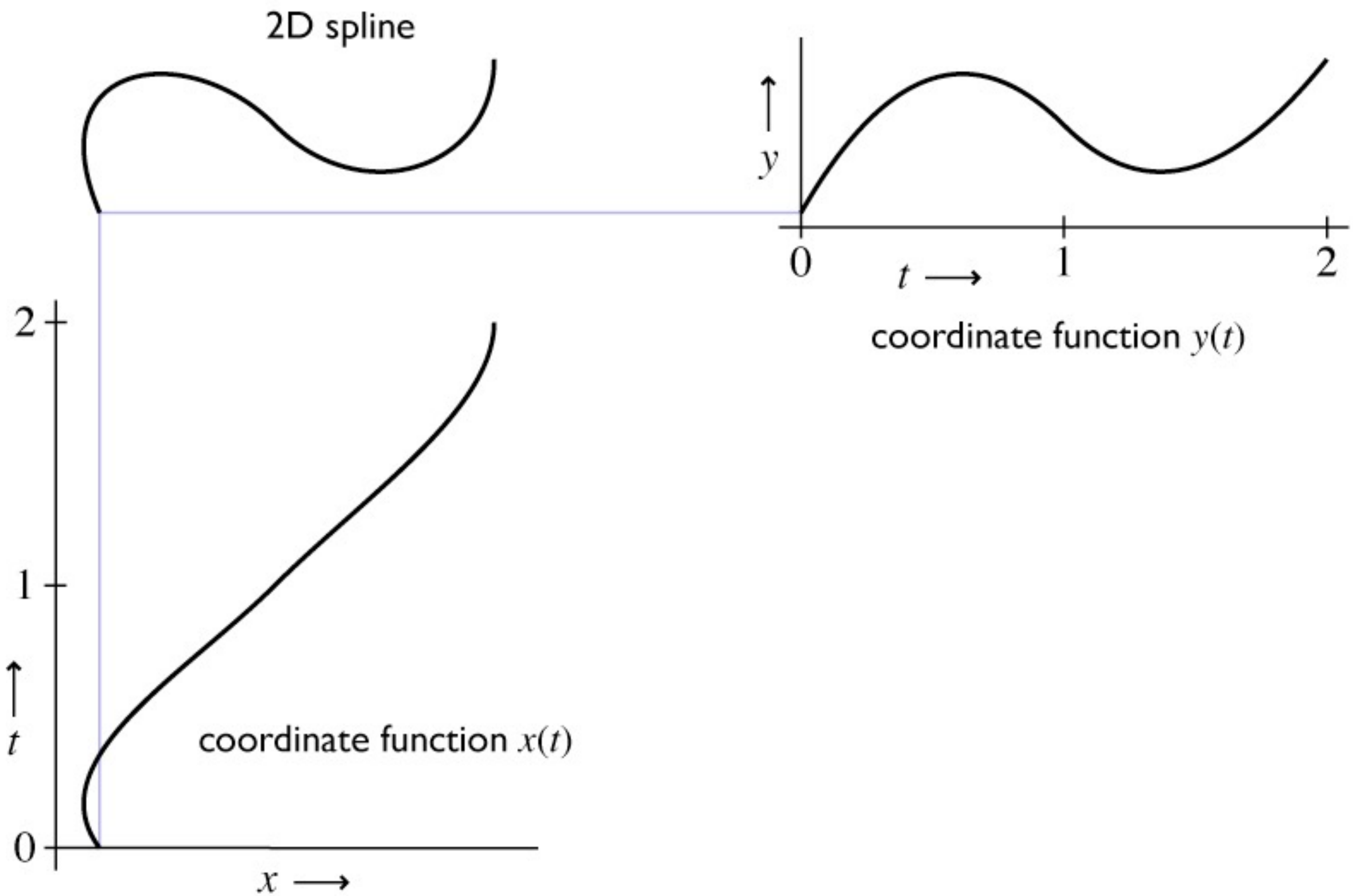

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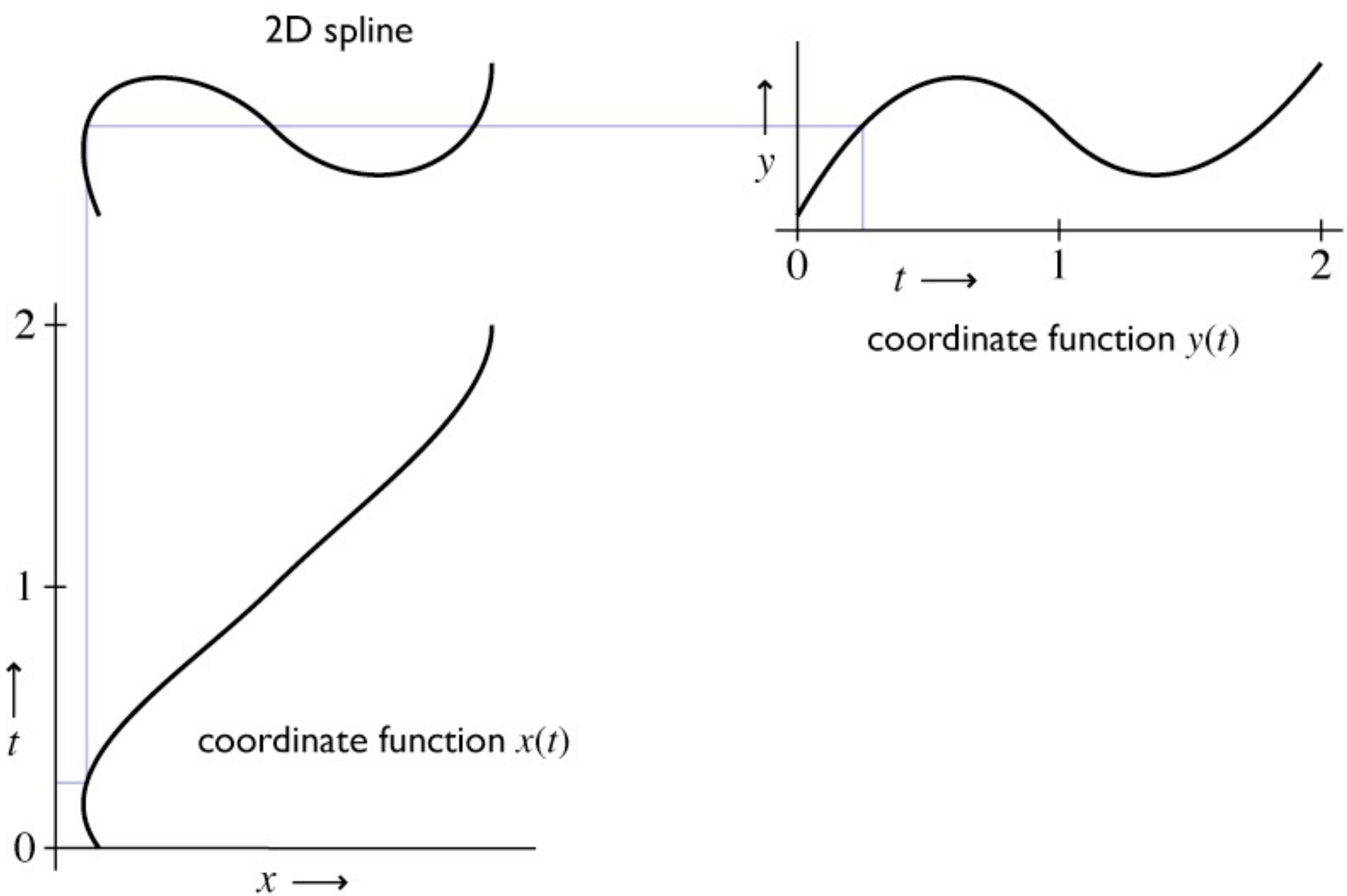

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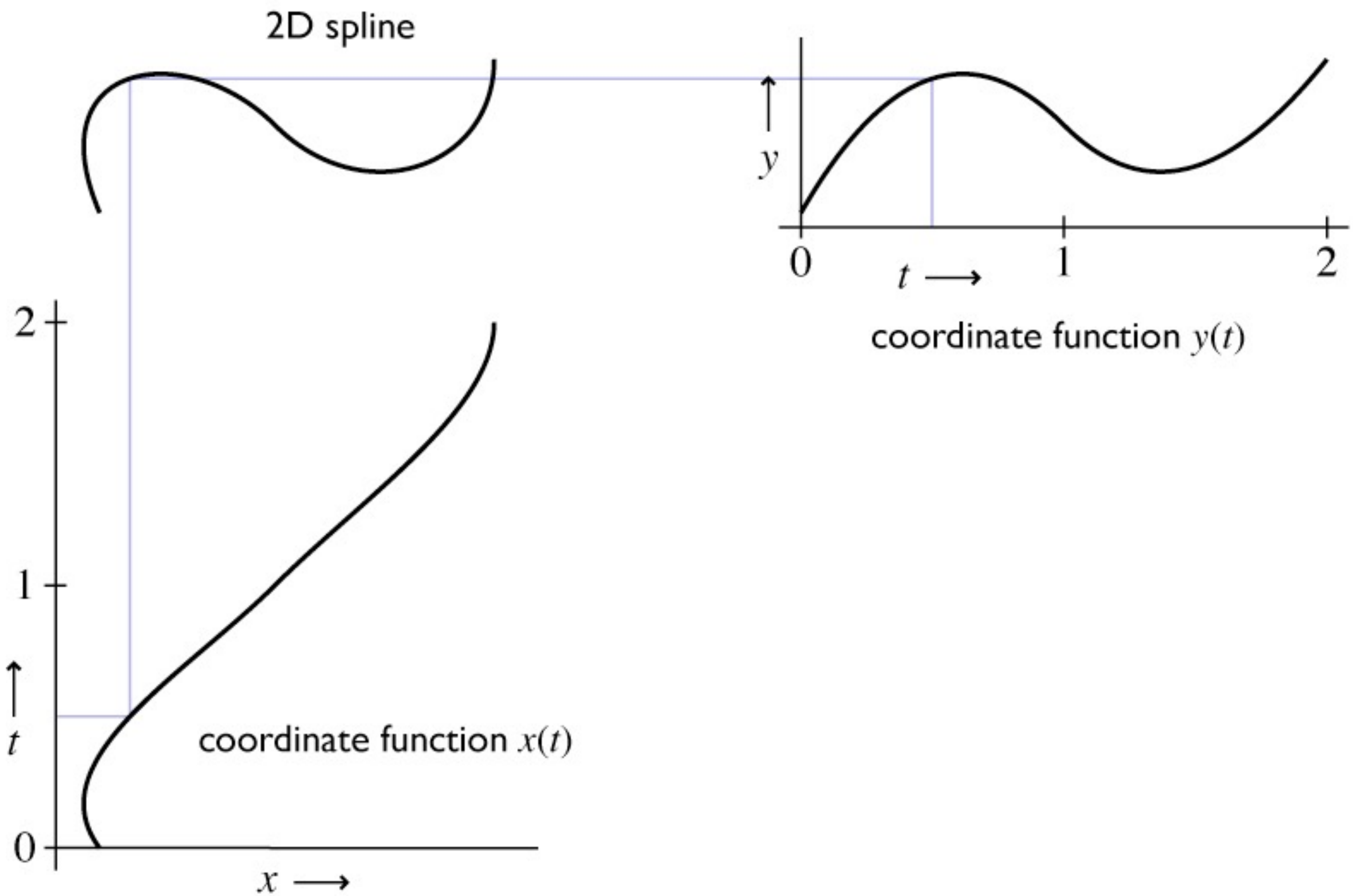

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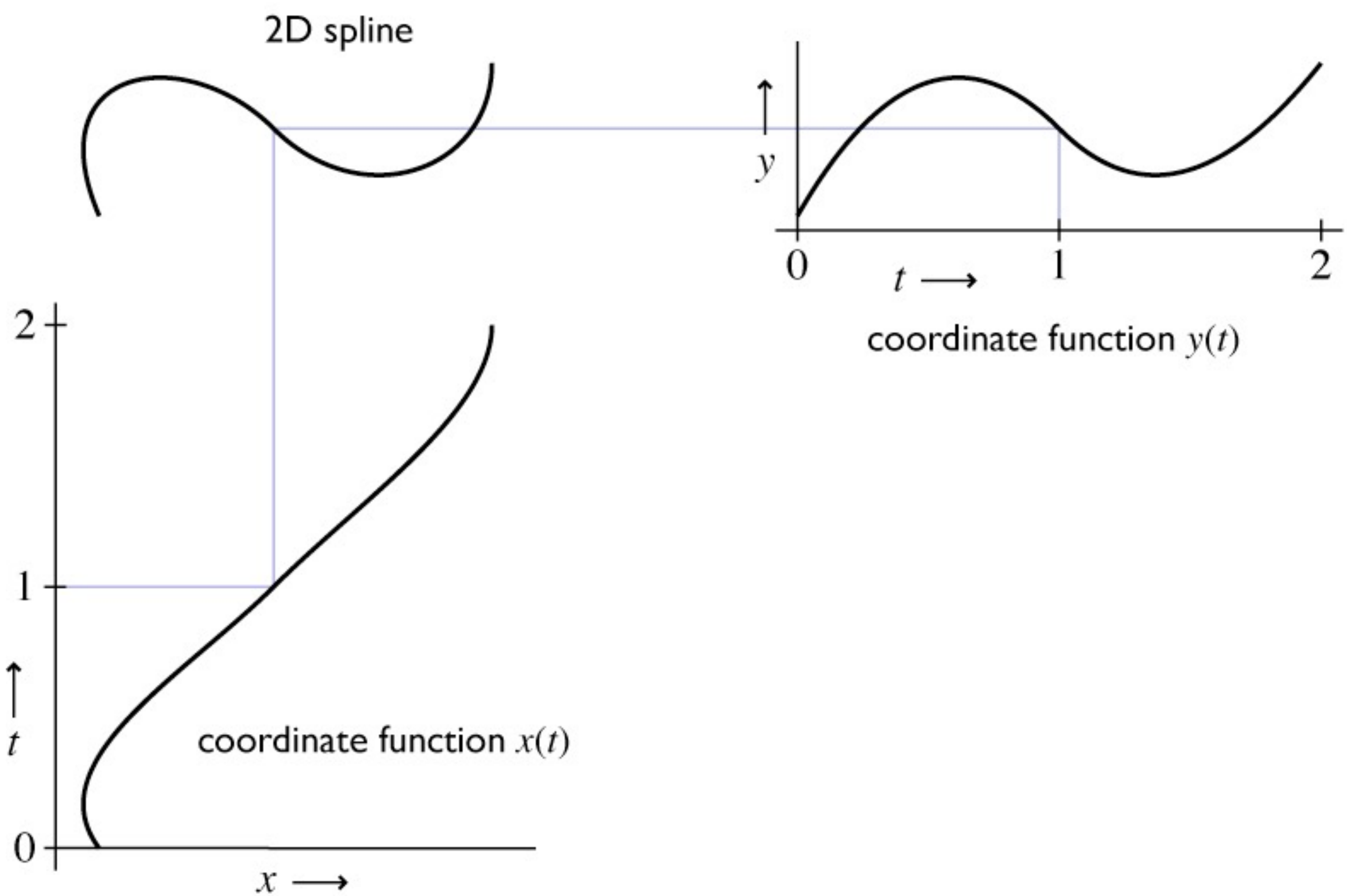

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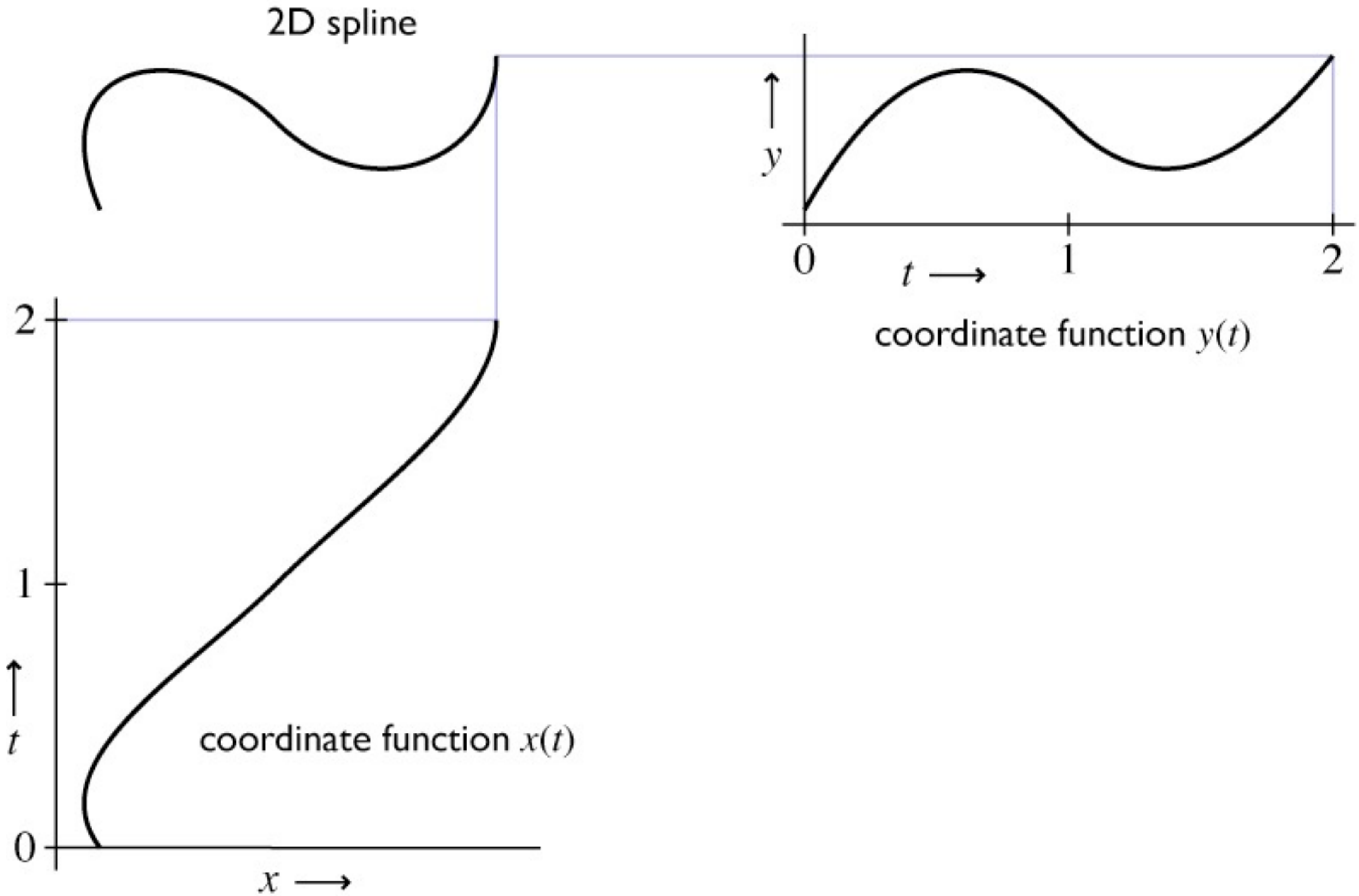
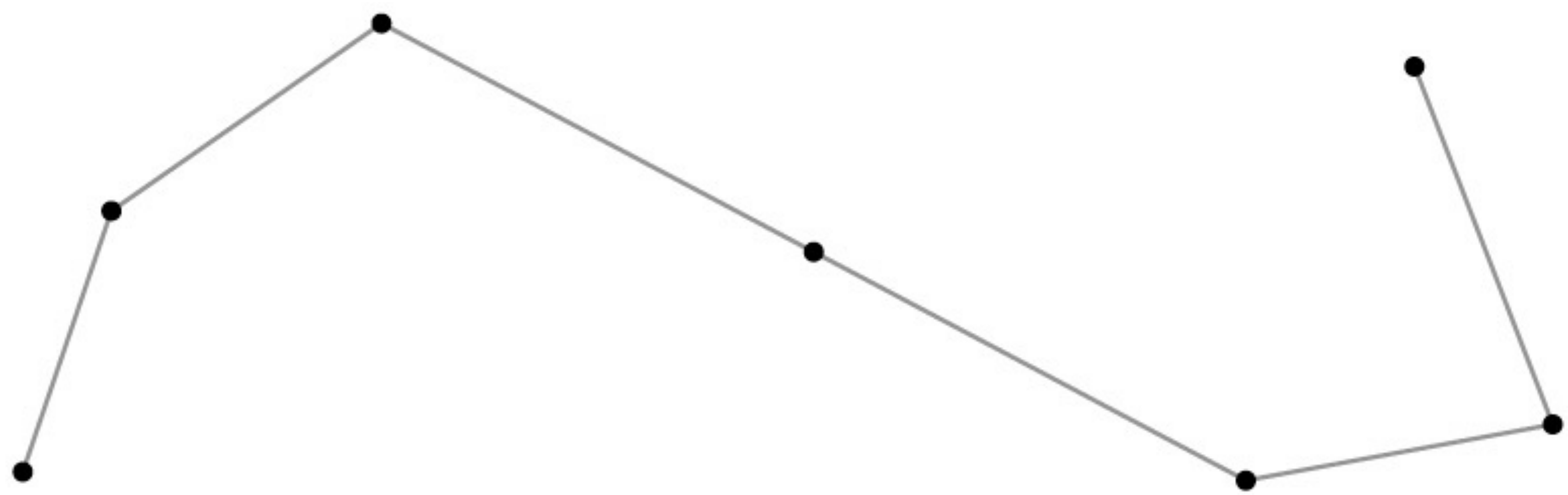


Control of spline curves

- Specified by a sequence of controls (points or vectors)
- Shape is guided by control points (aka control polygon)
  - interpolating: passes through points
  - approximating: merely guided by points



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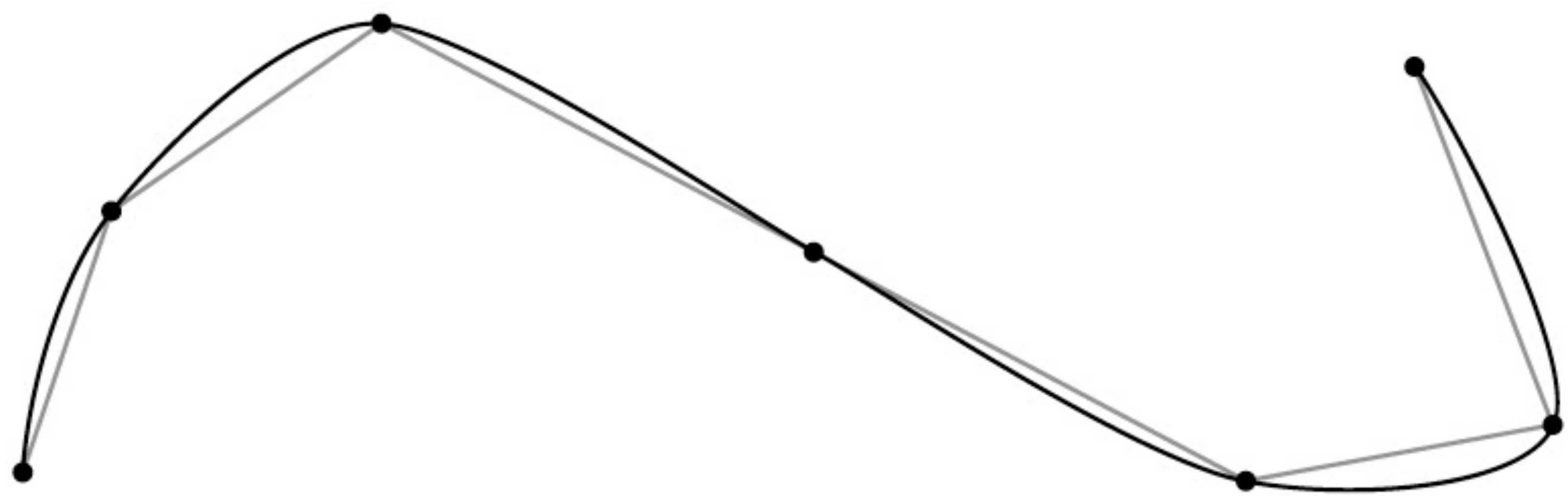

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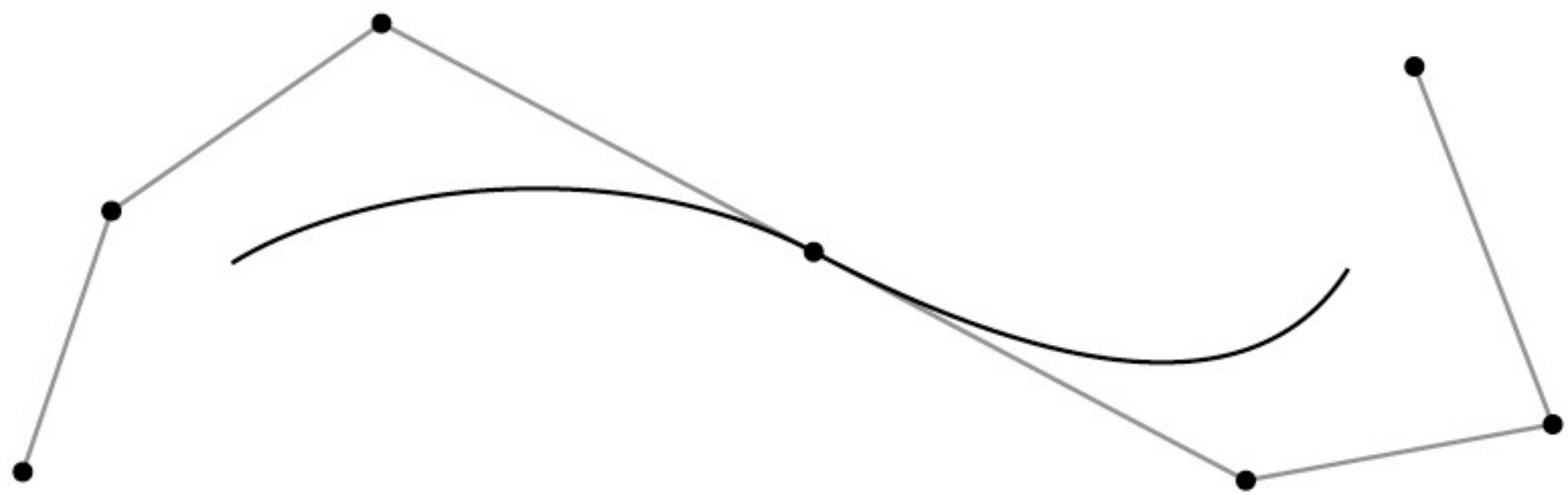

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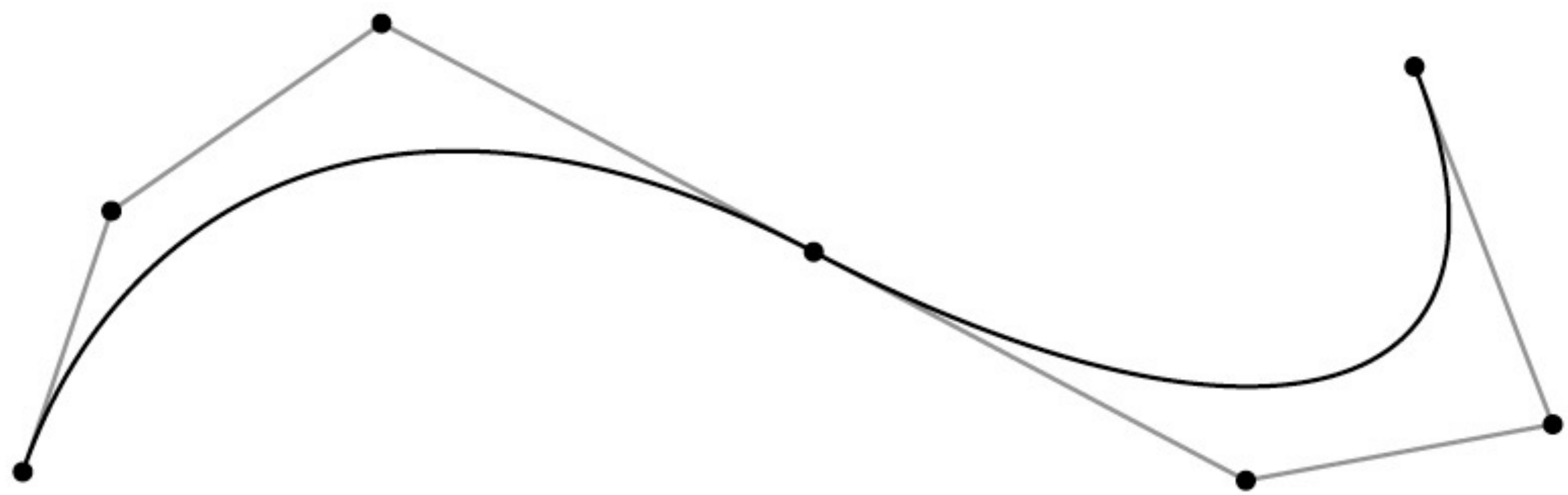


How splines depend on their controls

- Each coordinate is separate
  - the function $x(t)$ is determined solely by the x coordinates of the control points
  - this means 1D, 2D, 3D, … curves are all really the same

Plan

1. Spline segments
   - how to define a polynomial on [0,1]
   - …that has the properties you want
   - …and is easy to control
2. Spline curves
   - how to chain together lots of segments
   - …so that the whole curve has the properties you want
   - …and is easy to control
3. Refinement and evaluation
   - how to add detail to splines
   - how to approximate them with line segments

Spline Segments

Trivial example: piecewise linear

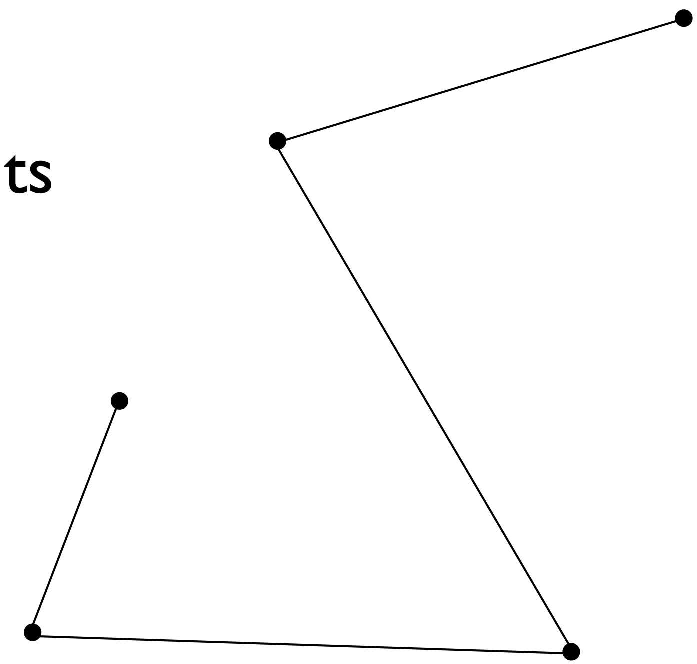
- This spline is just a polygon
  - control points are the vertices
- But we can derive it anyway as an illustration
- Each interval will be a linear function

– $x(t) = at + b$

  – constraints are values at endpoints

– $b = x_0$; $a = x_1 - x_0$

  – this is linear interpolation



Trivial example: piecewise linear

- Vector formulation

$$x(t) = (x_1 - x_0)t + x_0$$
$$y(t) = (y_1 - y_0)t + y_0$$
$$\mathbf{f}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0$$

- Matrix formulation

$$\mathbf{f}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

Trivial example: piecewise linear

- Basis function formulation
  - regroup expression by **p** rather than *t*

$$\begin{aligned}\mathbf{f}(t) &= (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0 \\ &= (1-t)\mathbf{p}_0 + t\mathbf{p}_1\end{aligned}$$

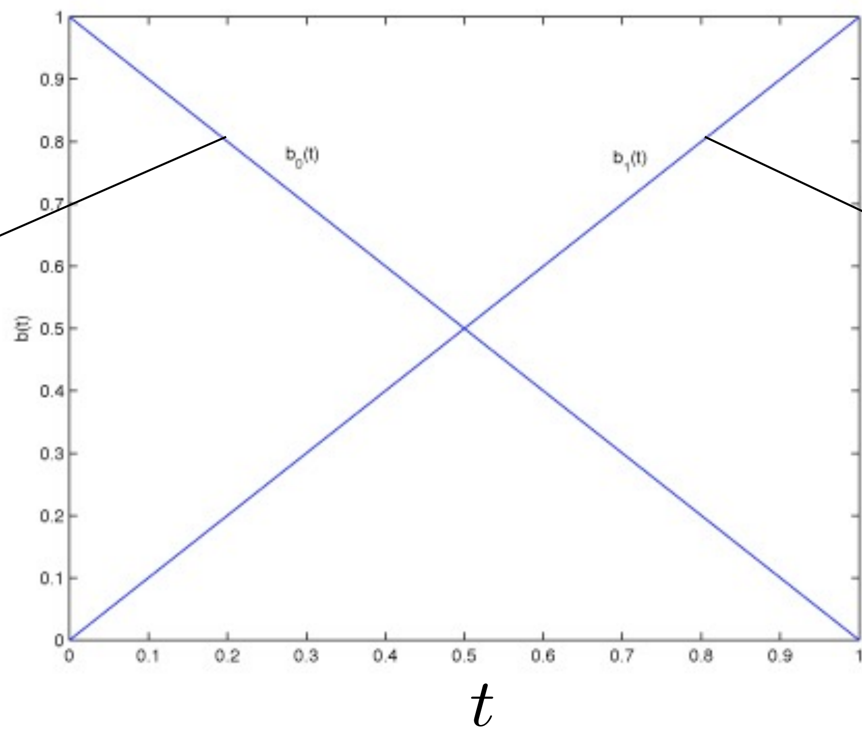
  - interpretation in matrix viewpoint

$$\mathbf{f}(t) = \left(\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix}$$

Trivial example: piecewise linear

- Vector blending formulation: "average of points"
  - blending functions: contribution of each point as *t* changes

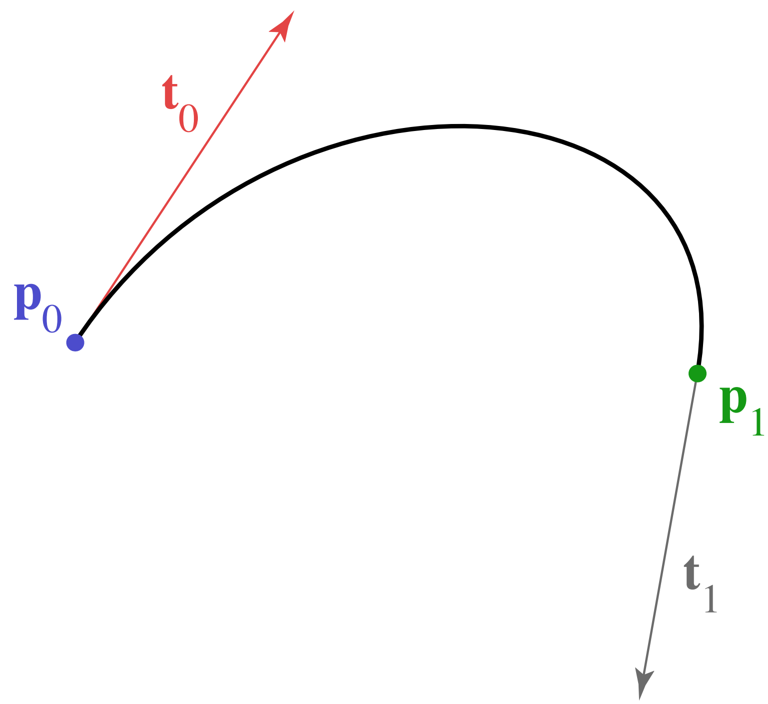
$$\mathbf{b}_0(t) = 1 - t$$



$$\mathbf{b}_1(t) = t$$

Hermite splines

- Less trivial example
- Form of curve: piecewise cubic
- Constraints: endpoints and tangents (derivatives)



Hermite splines

- Solve constraints to find coefficients

$$x(t) = at^3 + bt^2 + ct + d$$
$$x'(t) = 3at^2 + 2bt + c$$
$$x(0) = x_0 = d$$
$$x(1) = x_1 = a + b + c + d$$
$$x'(0) = x'_0 = c$$
$$x'(1) = x'_1 = 3a + 2b + c$$

$$d = x_0$$
$$c = x'_0$$
$$a = 2x_0 - 2x_1 + x'_0 + x'_1$$
$$b = -3x_0 + 3x_1 - 2x'_0 - x'_1$$

Matrix form of spline

$$\mathbf{f}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

$$\mathbf{f}(t) = b_0(t)\mathbf{p}_0 + b_1(t)\mathbf{p}_1 + b_2(t)\mathbf{p}_2 + b_3(t)\mathbf{p}_3$$

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Hermite splines

- Matrix form is much simpler

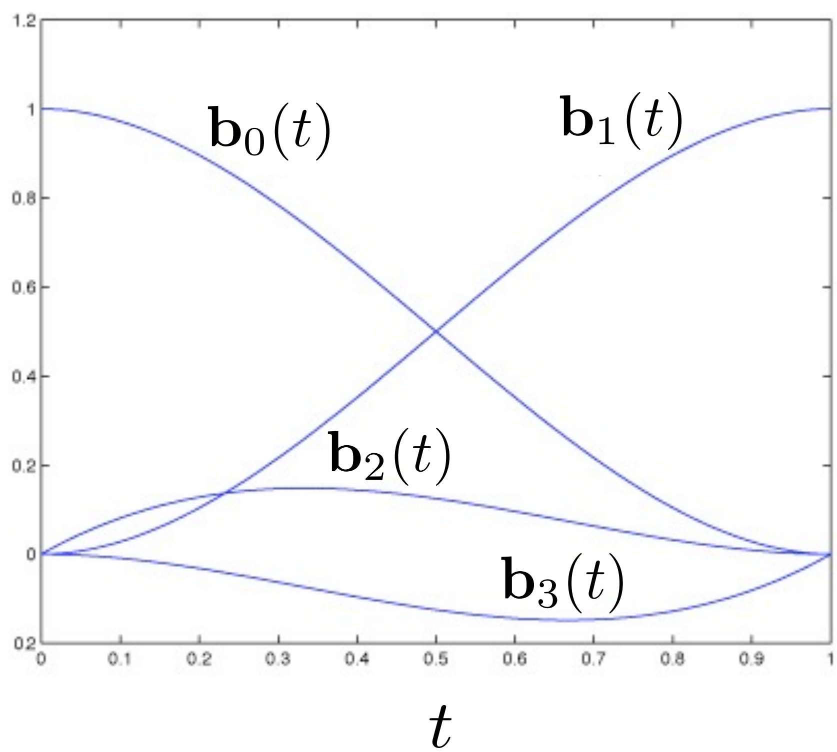
$$\mathbf{f}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{t}_0 \\ \mathbf{t}_1 \end{bmatrix}$$

  – coefficients = rows

  – basis functions = columns

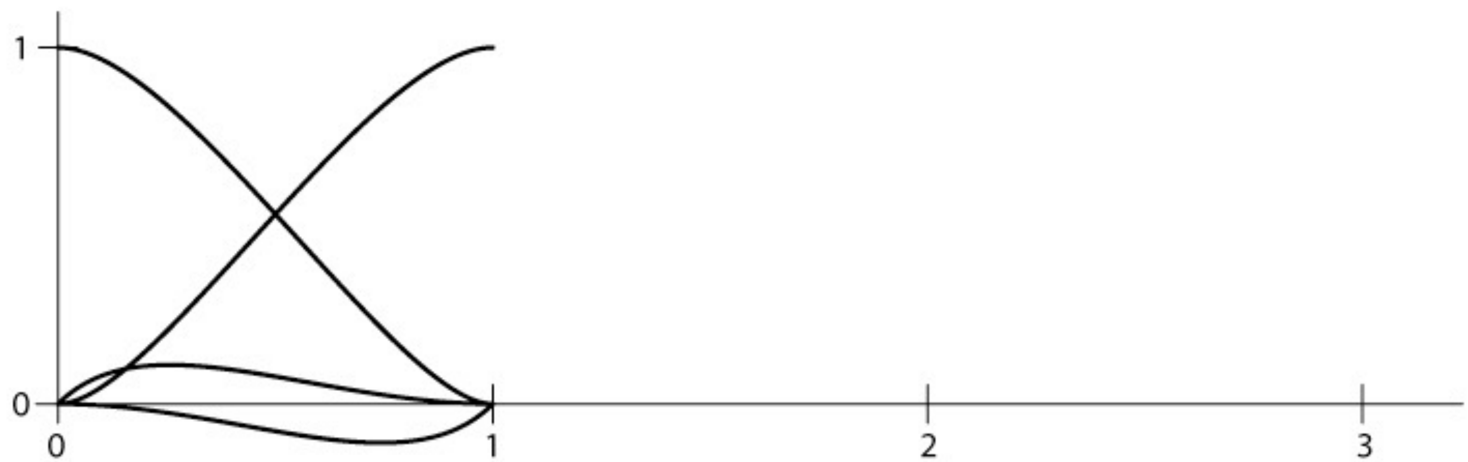
Hermite splines

- Hermite blending functions



Hermite splines

- Hermite basis functions



Hermite splines

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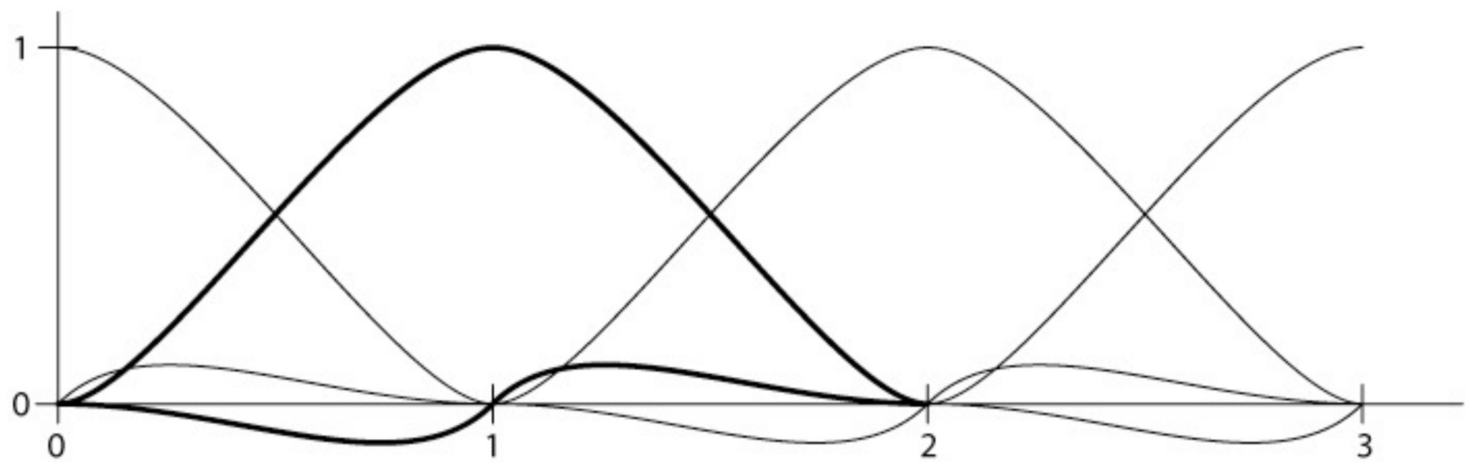
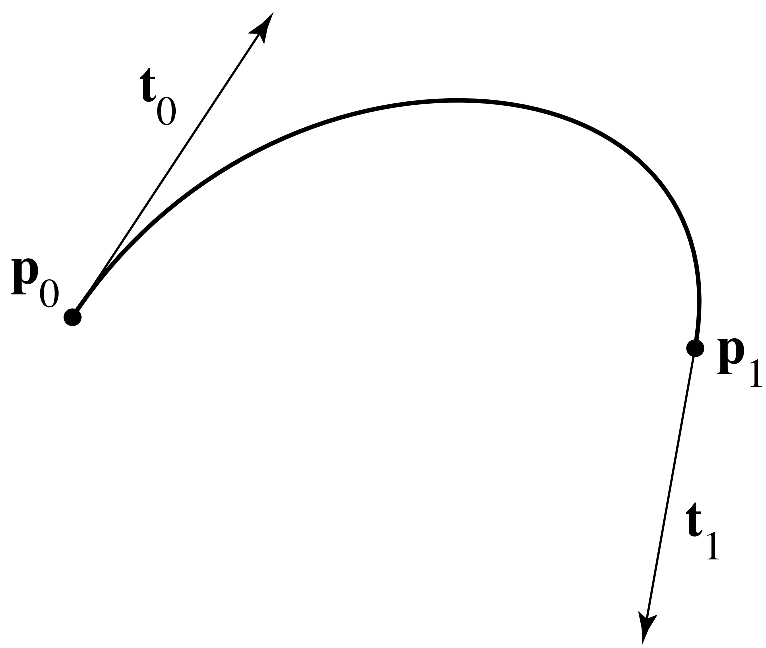


Hermite to Bézier

- Mixture of points and vectors is awkward
- Specify tangents as differences of points



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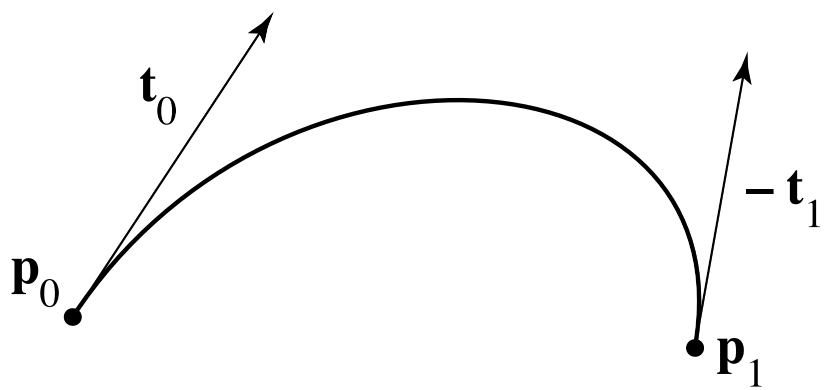

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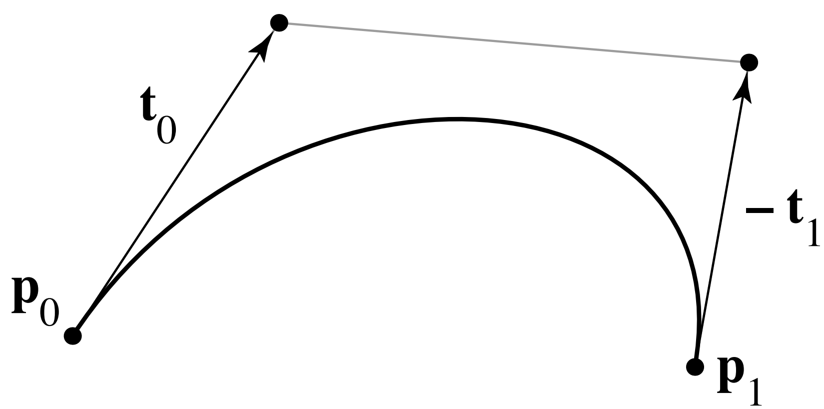

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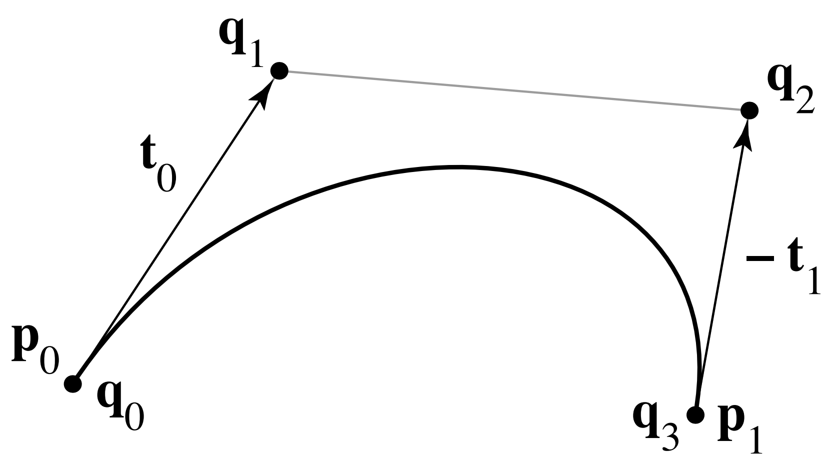
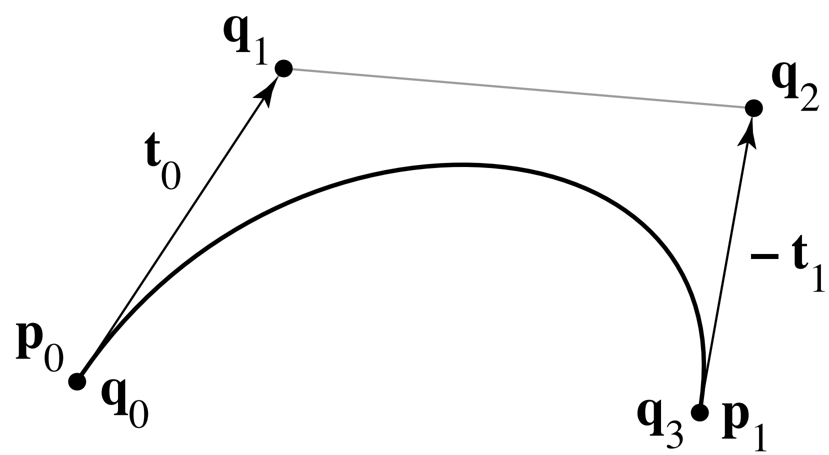


I'm calling these points **q** just for this slide and the next one.

– note derivative is defined as 3 times offset
  - reason is illustrated by linear case

Hermite to Bézier

$$\mathbf{p}_0 = \mathbf{q}_0$$
$$\mathbf{p}_1 = \mathbf{q}_3$$
$$\mathbf{t}_0 = 3(\mathbf{q}_1 - \mathbf{q}_0)$$
$$\mathbf{t}_1 = 3(\mathbf{q}_3 - \mathbf{q}_2)$$



$$\begin{bmatrix}\mathbf{p}_0\\\mathbf{p}_1\\\mathbf{v}_0\\\mathbf{v}_1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\-3 & 3 & 0 & 0\\0 & 0 & -3 & 3\end{bmatrix}\begin{bmatrix}\mathbf{q}_0\\\mathbf{q}_1\\\mathbf{q}_2\\\mathbf{q}_3\end{bmatrix}$$

Hermite to Bézier

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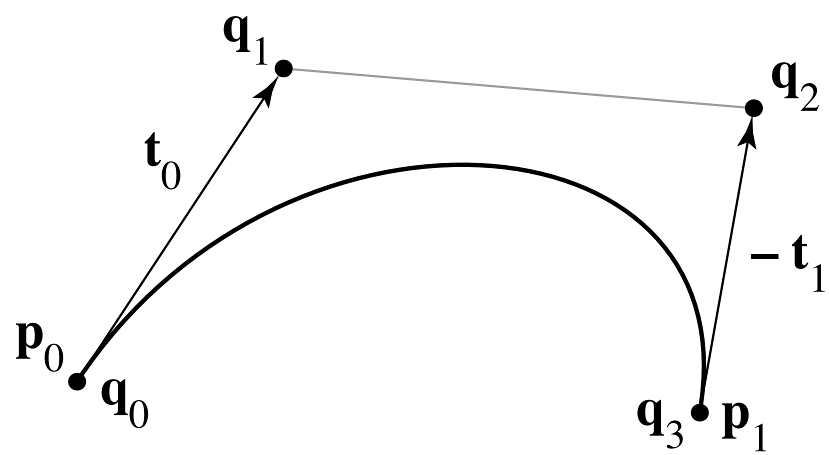

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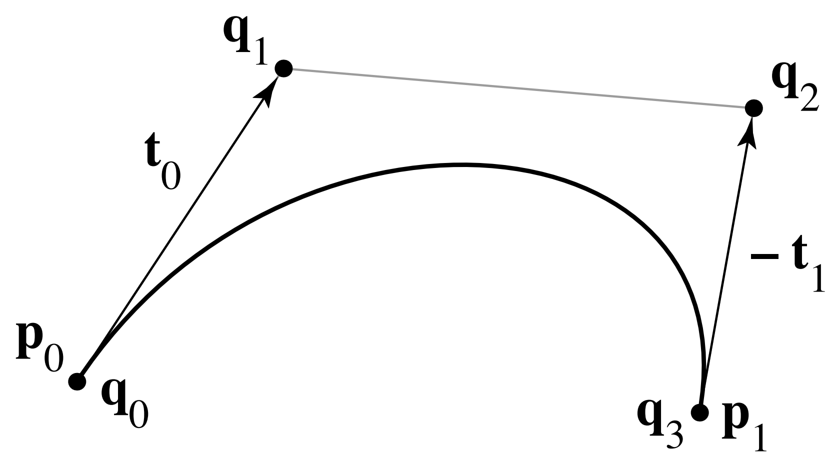
Hermite to Bézier



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$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

Bézier matrix

$$\mathbf{f}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

– note that these are the Bernstein polynomials

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

and that defines Bézier curves for any degree

Bézier basis