CS 421: Numerical Analysis
Fall 2005
Problem Set 6

Handed out: Mon., Nov. 21.

Due: Fri., Dec. 2 in lecture.

1. For both parts of this question, assume constant stepsize.

   (a) Determine the local truncation error (including the coefficient) of 2-step Adams Moulton rule, which is

   $$y_{k+1} = y_k + (5h/12)f(y_{k+1}, t_{k+1}) + (8h/12)f(y_k, t_k) - (h/12)f(y_{k-1}, t_{k-1})$$

   using the R-method from lecture.

   (b) Show that the order of accuracy of the method $y_{k+1} = 2y_k - y_{k-1}$ is 1 (i.e., same order as EM). There is something obviously wrong with this method! Explain why this method is completely useless in practice.

2. Consider a frictionless pendulum, whose equations of motion are:

   $$\begin{aligned} \frac{d\theta}{dt} &= v, \\ \frac{dv}{dt} &= -\sin\theta \end{aligned}$$

   where θ is the angle made by the pendulum with respect to vertical and v is the angular velocity of the pendulum.

   (a) Verify mathematically that this system conserves energy, where energy is $E = -\cos\theta + v^2/2$. [Hint: Compute dE/dt.]

   (b) Show that for Euler's method applied to the pendulum, $E_{k+1} = E_k + O(h^2)$. Here, E_k stands for $-\cos\theta_k + (v_k)^2/2$. Assuming the swings are small (θ_k fairly close to 0) and h is small, is $E_{k+1} - E_k$ positive or negative? [Hint: Expand $E_{k+1} - E_k$ as a Taylor series in h. Show that the $O(h)$ term vanishes, and study the coefficient of the $O(h^2)$ term to decide whether it will be positive or negative.]

3. It has been proposed in the literature to solve an unconstrained minimization problem of minimizing $f(\mathbf{x})$, where $f : \mathbf{R}^n \to \mathbf{R}$, by integrating the ODE given by $d\mathbf{x}/dt = -\nabla f(\mathbf{x})$. Note that if the trajectory defined by this ODE converges to a limit point $\mathbf{x}^*$ (i.e., if $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$, where $\mathbf{x}(t)$ satisfies the ODE), then the limit point is a a stationary point of the original optimization problem.

   (a) Suppose that forward Euler method is applied to this ODE. Assume $\nabla^2 f$ is positive definite. Relate the resulting iteration for integrating the ODE to a steepest descent method for optimizing f. Show also that this reformulation of steepest descent gives insight into selection of the line-search parameter in steepest descent. (Consider stability of Euler's method.)

(b) Suppose that the Backward Euler method is applied to this ODE, but that resulting system of nonlinear equations is solved only approximately by taking a single step of Newton's method (starting from the initial guess $\mathbf{x}_{k+1} = \mathbf{x}_k$) on each time step of the BE iteration. Relate the resulting iteration for integrating the ODE to a method for optimization that was mentioned in lecture.

4. Implement AB1 (i.e. Euler's method) and AB2 for the pendulum problem of Q2 in Matlab and track the energy of the system. Choose starting data $\theta_0 = \pi/4$ and $v_0 = 0$ (i.e., the pendulum is stationary and at a 45-degree angle). How well do each of them conserve energy for a fixed time step and fixed interval of integration? What happens to each when the time step is halved (but the interval of integration is fixed)? Note that you can initialize AB2 by taking one step of AB1 to start.

   Run AB1 for a very large number of steps. You will notice there is eventually a qualitative transition to a different kind of behavior. Can you explain this transition? (The same thing will happen to AB2, if the number of steps is large enough.)

   Turn in listings of your m-files, a paragraph or two of conclusions and at least one interesting plot.