CS 421: Numerical Analysis
Fall 2005
Problem Set 1

Handed out: Wed., Sep. 7.

Due: Fri., Sep. 16 in lecture.

1. Let M be an $n \times n$ *elementary* unit lower triangular matrix, that is, a matrix of the form $I - \mathbf{m}\mathbf{e}_k^T$ where $\mathbf{m} \in \mathbf{R}^n$ is a vector whose first k entries are 0's and $\mathbf{e}_k$ is the kth column of the identity matrix. See p. 67 of the text for an example and more explanation. Let $P(i,j)$ be the permutation matrix that exchanges row i with row j, but leaves other rows unchanged. Assume $i > k$ and $j > k$. Show that $P(i,j)M = NP(i,j)$, where N is some other elementary lower triangular matrix. Exactly how is N related to M?

2. Suppose the pivot in GEPP is zero. Prove that this implies that the original matrix is singular. Suggested approach: First, write down the form of the partially factored matrix that is the result of GEPP until the zero pivot is encountered; use block-matrix notation. Under the hypothesis that the pivot is zero, which entries of this matrix are zeros? Second, show that this matrix is singular by finding a nonzero vector in its nullspace. The argument is very similar to an argument made in lecture about the singularity of an upper triangular matrix with a 0 on the diagonal. Third, show that this partially factored matrix has the same rank as the original matrix by arguing that the rowspace of the partially factored matrix is the same as the rowspace of A.

3. Show that Heath's algorithm for GEPP applied to a singular matrix described on p. 71 can be modified to produce an infinite number of factorizations $PA = LU$, where P is a permutation matrix, A is a singular matrix, L is unit lower triangular with all multipliers in $[-1, 1]$, and U is upper triangular. The existence of an infinite number of factorizations may depend on which diagonal entry of U is zero.

   (Note: Heath explains on p. 78 that LU factorization of PA is unique, but his argument requires the assumption that PA is nonsingular.)

4. Construct a square random matrix in Matlab as follows: `A = randn(n,n);`. With extremely high probability, this matrix will be nonsingular: If you type `rank(A)`, you will get back the same value of n that you started with. It is a theorem from linear algebra that the product of two nonsingular square matrices is nonsingular, and therefore if A is nonsingular, so is A^2. By induction, so is A^k for any positive integer k. Does this hold in practice for Matlab? Generate such a matrix and try to find the value of k that makes A^k singular according to the `rank` function. Does k depend on n? The reason for the apparent decrease in rank arises from a combination of roundoff error and the behavior of power iteration, both of which will be covered later in the class. Hand in listings of all Matlab m-files, printouts or plots of any results you would like to highlight, any relevant traces of Matlab commands and their outputs, and some written comments that address the questions raised here.