CS 421: Numerical Analysis
Fall 2005
Prelim 2

Handed out: Friday., Nov. 4.

This exam has four questions. You have 72 hours to answer all questions. The questions are weighted equally even though they are not equally difficult. The exam counts for 20% of your final course grade (same as Prelim 1). This exam is picked up 2:15, Fri., Nov. 4 and due back 2:15, Mon., Nov. 7. Please pick up the exam in lecture and return your solutions in lecture.

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Heath and the lecture notes, then you must cite your sources.

You may also consult the web and other on-line resources. But you may not make any posting or send any email concerning the exam questions.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until the afternoon of Tues., Nov. 8 because some students may be handing in the exam late. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in up to 24 hours late will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are returned on time. No solutions will be accepted more than 24 hours late.

1. (The following question is related to certain recent algorithms for eigenvalue computation.) Let A be an $n \times n$ tridiagonal positive definite matrix.

   (a) Consider factoring A as LDL^T as in lecture. Determine the pattern of nonzero entries in L and D.

   (b) Define $\bar{A} = D^{1/2}L^TLD^{1/2}$. Show that $\bar{A}$ is also tridiagonal and symmetric.

   (c) Show that $\bar{A}$ is similar to A. [Hint: The similarity transform has an explicit formula in terms of L and D.]

2. Let A_1, A_2 be $m \times n$ and $p \times n$ matrices respectively, both of rank n. Suppose they are factored using the Householder algorithm $A_1 = Q_1R_1$ and $A_2 = Q_2R_2$. Develop an efficient algorithm for factoring $[A_1; A_2]$ as QR that uses Q_1, Q_2, R_1, R_2. Matrix Q may be represented implicitly, but please describe its representation. Analyze the number of flops required by your algorithm, accurate to the leading term.

3. Let H, J be two subspaces of $\mathbf{R}^n$ specified in implicit form. In other words, you are given two matrices A, B such that $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{p\times n}$ such that $H = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} = \mathbf{0}\}$ and $J = \{\mathbf{x} \in \mathbf{R}^n : B\mathbf{x} = \mathbf{0}\}$. Assume that $\text{rank}(A) = m$ and $\text{rank}(B) = p$. Propose an algorithm that takes as input A and B and returns a matrix C such that $K = \{\mathbf{x} : \mathbf{C}\mathbf{x} = \mathbf{0}\}$ contains both H and J and such that K is of minimal dimension (i.e., C has the maximum number of linearly independent rows) with this property. Your algorithm should involve a tolerance to account for roundoff error.

   [Hint: Apply QR factorization to A^T and B^T to produce parametric-form representations of H, J, say involving two matrices D, E with full column rank. Then apply the SVD to $[D, E]$ to determine the rank of $[D, E]$ and also a matrix C such that $CD = 0$ and $CE = 0$.]

4. Let A be a square invertible matrix. In Problem Set 2, Question 4, it was suggested that $\|A^{-1}\|_2$ could be estimated by selecting a random vector $\mathbf{x}$ and computing $\|A^{-1}\mathbf{x}\|_2/\|\mathbf{x}\|_2$. (The question pertained to the ∞-norm, but the results generalize to any p-norm.) In the same problem set, Question 2(b), it was shown that this estimate is always less than or equal to the true 2-norm of A^{-1}.

   (a) Let $\mathbf{x}$ be a nonzero vector. Show that $\|\mathbf{x}\|_2/\|A\mathbf{x}\|_2$ is also a lower bound on $\|A^{-1}\|_2$.

   (b) Consider the algorithm to estimate $\|A^{-1}\|_2$ that involves choosing several random vectors $\mathbf{x}$ and evaluating the quotient in part (a). Explain why this algorithm is expected to perform much worse (i.e., produce much poorer estimates) than the algorithm based on $\|A^{-1}\mathbf{x}\|_2/\|\mathbf{x}\|_2$, particularly when A is ill-conditioned.

   The suggested approach to part (b) is to use the SVD of A. A rigorous answer for part (b) is not required. Instead, use heuristic assumptions such as the following: if $\mathbf{x} \in \mathbf{R}^n$ is chosen randomly in a reasonable way (e.g., according to a uniform distribution on the unit sphere in $\mathbf{R}^n$), and if Q is any orthogonal matrix, then it is highly probably that the entries of $Q\mathbf{x}$ will all have roughly the same magnitude.