CS 421: Numerical Analysis
Fall 2005
Practice Prelim 1

Handed out: Thurs., Sep. 23.
This exam lasted 75 minutes. The students were allowed to consult a $8.5'' \times 11''$ piece of paper written on both sides. The questions were weighted equally even though they are not equally difficult.

1. Recall that the formula for the determinant of a 2×2 matrix A is $A(1,1)A(2,2) - A(1,2)A(2,1)$. Give an example of a 2×2 matrix such that computation of its determinant is very ill conditioned, i.e., a small relative change to A results in a large relative change to $\det(A)$. Given another matrix whose determinant is well conditioned. Be sure to explain your answers.

2. Consider solving the linear system
$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$
for $\mathbf{x}$ and $\mathbf{y}$ where A, B, C are given $n \times n$ matrices and $\mathbf{b}, \mathbf{c}$ are given n-vectors. Describe an efficient way to solve this system. For example, it may be preferable to carry out GEPP on parts of the matrix separately rather than GEPP on the whole matrix at once. How many flops (accurate to the leading term) are required for your algorithm?

3. Consider performing GEPP on an $n \times n$ matrix A that is upper triangular except for one additional nonzero entry in the $(n,1)$ position. How many flops are required for the factorization process (accurate to the leading term)? Does your answer depend on which entry (either $(1,1)$ or $(n,1)$) is selected as the first pivot?

4. Consider evaluating $(x-1)^{40}$ for various values of x in the interval $[0,2]$. One possible evaluation technique is to evaluate the formula directly. Another possibility is to evaluate the binomial expansion of the formula: $(x-1)^{40} = x^{40} - 40x^{39} + \binom{40}{2} x^{38} - \cdots$ Explain why the latter technique is unstable. Note: For your information, $\binom{40}{20} \approx 1.4 \cdot 10^{11}$.

5. For an $n \times n$ matrix A, let $\|A\|_* = \max_{i=1:n; j=1:n} |A(i,j)|$, that is, the maximum absolute entry of A.

   (a) Show that this definition satisfies the three axioms for a norm. Note: you can use the fact that the vector infinity-norm is already proven to satisfy the axioms.

   (b) Show via a counterexample that this norm is not submultiplicative. Hint: consider a matrix with many equal entries. A 2×2 counterexample will suffice.