CS 421: Numerical Analysis
Fall 2005
Practice Final Exam

Handed out: Tues., Nov. 29 (web only).

This exam had 11 questions and 120 points total. Students had 120 minutes to complete all the questions. This exam was closed-book and closed-note, but students could consult a prepared sheet of notes (one page, $8\frac{1}{2}'' \times 11''$ written on both sides).

1. [**5 points**] Given two vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ such that $\mathbf{v} \neq \mathbf{0}$, how many flops (accurate to the leading term) are required to compute $(I - \mathbf{v}\mathbf{v}^T/(\mathbf{v}^T\mathbf{v}))\mathbf{w}$?

2. [**5 points**] Same $\mathbf{v}$ as the previous question, and let W be an $n \times n$ matrix. How many flops (accurate to the leading term) are required to compute $(I - \mathbf{v}\mathbf{v}^T/(\mathbf{v}^T\mathbf{v}))W$?

3. [**5 points**] Let A be an $m \times n$ matrix with $m \geq n$. Is it possible to determine whether $\text{rank}(A) < n$ with Householder's QR factorization? If yes, briefly explain how. If no, briefly explain why not.

4. [**5 points**] The term "Fox Prize" could be a reference to the Leslie Fox Prize in Numerical Analysis. "Fox Prize" might also refer to the prize given to the best singer competing on the Fox TV show *American Idol.* Which Fox Prize would you rather win? [Note: +1 extra credit if you've already won either of these prizes. +2 if you've won both.]

5. [**10 points**] Let A be an $n \times n$ matrix, and let B be obtained from A by interchanging its first two rows. Explain why A and B have the same singular values.

6. [**10 points**] The function $f : \mathbf{R}^n \to \mathbf{R}$ given by $f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|_\infty$, where A is an $m \times n$ matrix and $\mathbf{b}$ is an n-vector, is an example of a *piecewise linear function.* Such a function has the property that there is a partition of $\mathbf{R}^n$ into a finite number of subsets $P_1, \ldots, P_s$ such that $f(\mathbf{x})$ is affine linear (i.e., has the form $\mathbf{a}^T\mathbf{x} + b$ for some $\mathbf{a}$ and b) on each subset. An even simpler example of a piecewise linear function is $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = |x|$. Explain why Newton's method for unconstrained minimization is useless for this class of objective functions.

7. [**10 points**] Let $\mathbf{v} \in \mathbf{R}^n$ be nonzero. The matrix $P = I - \mathbf{v}\mathbf{v}^T/(\mathbf{v}^T\mathbf{v})$ is an example of an "orthogonal projector". Show that it is symmetric and that its eigenvalues are exactly $1, 1, \ldots, 1, 0$. [Hint: extend $\mathbf{v}/\|\mathbf{v}\|_2$ to an orthogonal basis. These are the eigenvectors.]

8. [**20 points**] Consider the finite difference method for IVP's given by

$$y_{k+1} = \alpha y_k + \beta y_{k-1} + \gamma h f(y_{k+1}, t_{k+1}).$$

(a) Is this method implicit or explicit?

(b) Determine values of α, β and γ in order to make this method second order. Assume constant stepsize h.

9. **[15 points]** Let C be a given $n \times n$ matrix and let A be a given $n \times 2$ matrix. Consider the problem of finding $X \in \mathbf{R}^{2\times n}$ such that $\|AX - C\|_F$ is minimized. Show that this problem can be rewritten as linear least squares for the entries of X by exhibiting the coefficient matrix and right-hand side of a standard-form linear least-squares problem. (Recall that $\|\cdot\|_F$ denotes the Frobenius norm, that is, the square root of sum of squares of matrix entries.)

10. **[20 points]** Consider the problem of diagonalizing a 2×2 symmetric matrix $A = [a, b; b, d]$. This can be posed as finding a 2×2 orthogonal matrix of the form $Q = [c, s; -s, c]$ such that $c^2 + s^2 = 1$ such that the $(2, 1)$ entry of $Q^T AQ$ is zero.

    (a) If the $(2, 1)$ entry of $Q^T AQ$ is zero, then $Q^T AQ$ is necessarily diagonal. Why?

    (b) Write down a system of two independent nonlinear equations that (c, s) must satisfy in order for Q to diagonalize A, and then write down a Newton method to solve these equations (in particular, be sure to write down the Jacobian of those equations). The equations will probably involve a, b, d in the coefficients.

11. **[15 points]** Consider the initial value problem $d\mathbf{y}/dt = f(\mathbf{y}, t)$, $\mathbf{y}(0) = \mathbf{y}_0$, where $\mathbf{y}(t) \in \mathbf{R}^n$. Suppose there exists a nonzero $\mathbf{v} \in \mathbf{R}^n$ such that $\mathbf{v}^T f(\mathbf{y}, t) = 0$ for all $\mathbf{y}$ and all t.

    (a) Show that for the true solution $\mathbf{y}(t)$ to the IVP, $\mathbf{v}^T\mathbf{y}(t)$ does not depend on t.

    (b) Show that for either the backward and forward Euler methods applied to this problem, $\mathbf{v}^T\mathbf{y}_k$ does not depend on k.