CS 421: Numerical Analysis
Fall 2000
Problem Set 6

Handed out: Mon., Nov. 20.

Due: Fri., Dec. 1 in lecture.

1. An integration method for an IVP $dy/dt = f(y,t)$, $y(0) = y_0$ is said to be "exact" if $y_k = y(t_k)$ for all values of k. Consider the IVP given by $dy/dt = t^l$, $y(0) = 0$, where l is a nonnegative integer. Note that the true solution to this IVP is $y(t) = t^{l+1}/(l+1)$. For both Euler and AB2, constant stepsize h, determine whether they are exact for these problems in the cases $l = 0, 1, 2$. In the case of AB2, assume that y_{-1} is exactly known for initialization.

2. Consider the the *implicit midpoint (IM) rule* for integrating an ODE. This rule is defined by the formula:

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h_k f((\mathbf{y}_k + \mathbf{y}_{k+1})/2, (t_k + t_{k+1})/2).$$

Consider applying this rule to the IVP

$$\begin{aligned} d\mathbf{y}/dt &= A\mathbf{y}, \\ \mathbf{y}(0) &= \mathbf{y}_0, \end{aligned}$$

where A is an $n \times n$ matrix. Show that if all the eigenvalues of A have negative real parts, then $\mathbf{y}_k$ computed by IM will tend to zero as $k \to \infty$. Assume a fixed time-step (i.e., $h_k = h$ independent of k). [Hint: The matrix $(I + \alpha A)^{-1}(I + \beta A)$ has the same eigenvectors as A. Why? How are its eigenvalues related to those of A?]

3. Consider a frictionless pendulum, whose equations of motion are:

$$\begin{aligned} \frac{d\theta}{dt} &= v, \\ \frac{dv}{dt} &= -\sin\theta \end{aligned}$$

where θ is the angle made by the pendulum with respect to vertical and v is the angular velocity of the pendulum.

(a) Verify mathematically that this system conserves energy, where energy is $E = -\cos\theta + v^2/2$. [Hint: Compute dE/dt.]

(b) Show that for Euler's method applied to the pendulum, $E_{k+1} = E_k + O(h^2)$. Assuming the swings are small (θ fairly close to 0) and h is small, is $E_{k+1} - E_k$ positive or negative? [Hint: Expand $E_{k+1} - E_k$ as a Taylor series. Show that the $O(h)$ term vanishes, and study the coefficient of the $O(h^2)$ term to decide whether it will be positive or negative.]

4. Implement AB1 (i.e. Euler's method) and AB2 for the pendulum problem of Q3 in Matlab and track the energy of the system. Choose starting data $\theta_0 = \pi/4$ and $v_0 = 0$ (i.e., the pendulum is stationary and horizontal). How well do each of them conserve energy for a fixed time step and fixed interval of integration? What happens to each when the time step is halved (but the interval of integration is fixed)? Note that you can initialize AB2 by taking one step of AB1 to start.

   Run AB1 for a very large number of steps. You will notice there is eventually a qualitative transition to a different kind of behavior. Can you explain this transition? (The same thing will happen to AB2, if the number of steps is large enough.)

   Turn in listings of your m-files, a paragraph or two of conclusions and at least one interesting plot.