CS 421: Numerical Analysis
Fall 2000
Problem Set 4

Handed out: Wed., Oct. 18.

Due: Fri., Oct. 27 in lecture.

1. (a) Show that the eigenvalues of an upper triangular matrix $U \in \mathbf{R}^{n\times n}$ are exactly the diagonal entries of U. [Hint: Which upper triangular matrices are singular?]

   (b) Propose an algorithm that takes as input an upper triangular matrix $U \in \mathbf{R}^{n\times n}$ with distinct diagonal entries and an index $i \in \{1,\ldots,n\}$ and computes an eigenvector $\mathbf{x}$ such that $U\mathbf{x} = U(i,i)\mathbf{x}$. [Hint: try back substitution.]

2. Consider applying orthogonal iteration with $l = n$ to an $n \times n$ nonsingular symmetric matrix A. Show that the nth column of $X^{(k)}$ is actually undergoing the inverse power method. [Hint: apply the $-T$ operation (inverse transpose) to the equations defining orthogonal iteration.]

3. The *Rayleigh quotient iteration* (RQI) is a method for finding an eigenvector. RQI is similar to the inverse shifted power method, except that the shift is recomputed on each iteration to be the Rayleigh quotient. Thus, the kth iteration of the RQI is:

$$\begin{aligned} \mathbf{x}^{(k+1)} &= (A - \sigma^{(k)} I)^{-1}\mathbf{x}^{(k)}, \\ \sigma^{(k+1)} &= (\mathbf{x}^{(k+1)})^T A \mathbf{x}^{(k+1)} / ((\mathbf{x}^{(k)})^T \mathbf{x}^{(k)}). \end{aligned}$$

   A difficulty with the RQI is that each iteration apparently requires $O(n^3)$ flops (as opposed to $O(n^2)$ flops for the inverse shifted power method) because the matrix changes on each iteration, so a new factorization is needed. Explain how to implement the RQI so that it requires only $O(n)$ flops per iteration by factoring $A = QTQ^T$ in a preliminary step.

4. The *matrix exponential* is described in Computer Problem 4.13 of Heath (p. 147), and also in 11.3 of GVL3. This operation is important for solving linear ordinary differential equations. Examine the three functions `expm1`, `expm2` and `expm3` in Matlab, all of which compute the matrix exponential, to see how they work. (Note: use the `type` command to print out m-files on your screen.)

   Come up with a matrix such that `expm1`, `expm2` and `expm3` all give the same (or very close) answers. Then come up with a matrix where `expm2` works poorly compared to `expm3` and `expm1`. [Hint: use the material from lecture about when the Taylor series for $\exp(x)$ is unstable.] Finally, come up with a matrix where `expm3` works poorly compared to `expm1` and `expm2`. [Hint: if A is nondiagonalizable, then its eigenvectors are infinitely ill-conditioned.]

Hand in: listings of all m-files that you wrote (note: you may not need to write m-files for this problem), traces of sample runs and printouts that answer the question.

The classic reference on this problem is:

C. Moler and C. Van Loan, "Nineteen dubious ways to compute the matrix exponential," SIAM Review (1978) 20:801-836.