CS 421: Numerical Analysis
Fall 2000
Prelim 2

Handed out: Friday., Nov. 3.

This exam has four questions. You have 72 hours to answer all questions. The questions are weighted equally even though they are not equally difficult. The exam counts for 20% of your final course grade (same as Prelim 1). There are two possible periods for this exam:

- 2:15, Fri., Nov. 3 until 2:15, Mon., Nov. 6. Please pick up the exam in lecture and return your solutions in lecture.

- 2:15, Mon., Nov. 6 until 2:15, Thurs., Nov. 9. Please pick up the exam in lecture and return your solutions to 493 Rhodes (Vavasis' office).

The exam is open-book and open-note. You may also consult outside published sources. If you use material from sources other than Heath and the lecture notes, then you must cite your sources.

You may also consult the web and other on-line resources. But you may not make any posting or send any email concerning the exam questions.

Academic integrity. You are not allowed to collaborate at all on this exam. You should refrain from bringing up the questions in any kind of discussion until the afternoon of Fri., Nov. 10 because some students may be handing in the exam late. The following kind of cooperation is allowed: You are permitted to borrow and photocopy someone else's lecture notes or other publicly-available course material. Please write and sign the following statement on your solutions: "I have neither given nor received unpermitted assistance on this exam."

Help from the instructor. The only help available will be clarification of the questions. No help will be given towards finding a solution.

Late acceptance policy. Solutions turned in up to 24 hours late will be accepted with a 10% penalty. The full late penalty is applied even if a portion of the solutions are returned on time. No solutions will be accepted more than 24 hours late.

1. Let $A \in \mathbf{R}^{n\times n}$ be a symmetric positive semidefinite matrix, and suppose plain Gaussian elimination (no pivoting) is performed on A. Suppose that a zero pivot is encountered on step k, whereas all previous pivots were nonzero. Show that the entire first row and first column of the uneliminated portion must be all zeros.

2. Let P be an orthogonal projection as in PS3, that is, a matrix of the form $P = A(A^TA)^{-1}A^T$ where A is an $m \times n$ matrix of rank n. Show that P is a symmetric matrix, all of whose eigenvalues are 1 or 0. Conversely, show that if P is an $m \times m$ symmetric matrix all of whose eigenvalues are 1 or 0, then there exists an $m \times n$ matrix A of rank n such that $P = A(A^TA)^{-1}A^T$. Here, n stands for the number of eigenvalues that are 1's.

3. Let $A \in \mathbf{R}^{n\times n}$ be a nonsingular symmetric matrix. Consider the following algorithm.

   $\tilde{X}^{(0)}$ = random $n \times l$ matrix of rank l;
   for $k = 0 : s-1$;
   $\quad\tilde{X}^{(k+1)} = A\tilde{X}^{(k)}$;
   end
   Factor $\tilde{X}^{(s)}$ as $X^{(s)}R^{(s)}$ (reduced QR factorization).

   Argue that this algorithm, in exact arithmetic, computes the same $X^{(s)}$ as orthogonal iteration, except possibly for signs.

   [Hint: You can use the following fact without proof: the factors Q and R of a reduced QR-factorization of an $m \times n$ matrix of rank n are uniquely determined, except possibly the signs of columns of Q may be changed, and the signs of rows of R are changed correspondingly. Note: The algorithm in this question is not useful in practice because it is unstable.]

4. (a) Derive an algorithm to solve the following problem. Given $A \in \mathbf{R}^{m\times n}$ find $\mathbf{x} \in \mathbf{R}^m$ and $\mathbf{y} \in \mathbf{R}^n$ to maximize $\mathbf{x}^TA\mathbf{y}$, subject to the constraints that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$. In fact, show also that the maximum value of $\mathbf{x}^TA\mathbf{y}$ under these constraints is equal to $\|A\|_2$.

   [Hint for the algorithm: use the SVD. Hint for showing that $\|A\|_2 = \sigma_1(A)$ is the maximum possible value: use the Cauchy-Schwarz inequality.]

   (b) The following problem arises in structural mechanics. Let $A \in \mathbf{R}^{n\times n}$ be symmetric. The problem is to find $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ that maximize $\mathbf{x}^TA\mathbf{y}$ subject to the constraints that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, and $\mathbf{x}^T\mathbf{y} = 0$. Develop an algorithm to solve this problem, and justify the steps of the algorithm.

   [Hints: (1) First solve the problem in the very special case that A is 2×2 of the form $A = \mathrm{diag}(a, -a)$. You can use part (a) of this question to argue that the solution you found is optimal. (2) Then generalize to the case of an $n \times n$ diagonal matrix whose maximum diagonal entry is a and whose minimum diagonal entry is $-a$. (3) Then generalize to an arbitrary $n \times n$ diagonal matrix by claiming that the optimal $\mathbf{x}, \mathbf{y}$ are unchanged if A is replaced by $A + \alpha I$. (4) Finally, use diagonalization and change of variables to solve for the general case.]