CS 421: Numerical Analysis
Fall 2000
Practice Final Exam

Handed out: Wed., Nov. 29.

This exam had 13 questions and 120 points total. The students had 120 minutes to complete all the questions. This exam was closed-book and closed-note, but students could consult a prepared sheet of notes (one page, $8\frac{1}{2}'' \times 11''$ written on both sides).

The following questions are short answer—a single phrase or formula suffices.

1. [**5 points**] Give a reason why Cholesky factorization is preferable to Gaussian elimination with partial pivoting (GEPP) for solving symmetric positive definite linear systems.

2. [**5 points**] Give an example of an unstable algorithm.

3. [**5 points**] How many flops (accurate to the leading term) are required to form $A^T A$, where A is an $m \times n$ matrix?

4. [**5 points**] Newton's method for solving nonlinear equations doesn't always converge. Name something that could go wrong with the method to prevent or hinder convergence.

5. [**5 points**] Consider a finite-difference method for integrating an initial value problem with time-step h. Suppose that on the kth step, the truncation error introduced on that step (i.e., disregarding errors from previous steps) is of the form $y'''(t_k)h^3/12$. What is the order of this method?

6. [**5 points**] Write down a 2×2 rank-one matrix.

7. [**5 points**] In the inverse shifted power method, what is a desirable property of the shift in order to attain fast convergence?

8. [**5 points**] Bill Clinton, the U.S. President, is not a numerical analyst. Can you name any other famous person who is not a numerical analyst?

These questions require longer answers.

9. [**15 points**] Let $A \in \mathbf{R}^{m\times n}$ be factored as QR, where $Q \in \mathbf{R}^{m\times m}$ is orthogonal and $R \in \mathbf{R}^{m\times n}$ is upper triangular. Assume A has full rank. Show that A and $Q(:,1:n)$ have the same range-space, and in fact, come up with an algorithm to solve $A\mathbf{x} = Q(:,1:n)\mathbf{y}$ for $\mathbf{x}$ given $\mathbf{y}$. Come up with a second algorithm that solves for $\mathbf{y}$ given $\mathbf{x}$.

   [Note: $Q(:,1:n)$ means the submatrix of Q formed by its first n columns.]

10. [**15 points**] Given $A \in \mathbf{R}^{n\times n}$ and a nonzero vector $\mathbf{y} \in \mathbf{R}^n$, suppose the statement

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y = A * y(n:-1:1);
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   is executed repeatedly in Matlab. Clearly this is some kind of a power method. What would you expect it to converge to (after normalization)?

   [Note: In case you are not familiar with Matlab notation, the above statement means: set $\mathbf{y}$ to the product $A\mathbf{z}$, where $\mathbf{z}$ is defined as the vector whose entries are the entries in $\mathbf{y}$ in reverse order.]

11. [**15 points**] Describe the tradeoffs involved in using an explicit versus implicit method for integrating an initial value problem. Under what circumstances does one or the other have a clear advantage?

12. [**15 points**] Let $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ have the property that, for each $i = 1, \ldots, n$, f_i depends only on $x_i, \ldots, x_n$ (and not on $x_1, \ldots, x_{i-1}$). Show that the Newton step for solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ can be computed especially efficiently. How many flops (accurate to the leading term) are required to compute the Newton step (once the Jacobian is known)?

13. [**20 points**] Consider the the *implicit midpoint (IM) rule* for integrating an ODE. This rule is defined by the formula:

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h_k f((\mathbf{y}_k + \mathbf{y}_{k+1})/2, (t_k + t_{k+1})/2).$$

   Consider applying this rule to the IVP

$$\begin{aligned} d\mathbf{y}/dt &= A\mathbf{y}, \\ \mathbf{y}(0) &= \mathbf{y}_0, \end{aligned}$$

   where A is an $n \times n$ matrix. Show that if all the eigenvalues of A have negative real parts, then $\mathbf{y}_k$ computed by IM will tend to zero as $k \to \infty$. Assume a fixed time-step (i.e., $h_k = h$ independent of k). [Hint: The matrix $(I + \alpha A)^{-1}(I + \beta A)$ has the same eigenvectors as A. Why? How are its eigenvalues related to those of A?]