2D Geometry and Transformations

CS 417 Lecture 7

Introduction

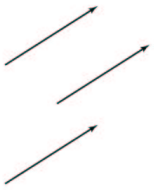
- So far: Imaging
- Next: Modeling
  - the study of how to define and manipulate the shapes and arrangements of objects
- We'll start with the 2D case, then later move to the more complex 3D case

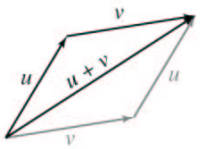
Mathematical preliminaries

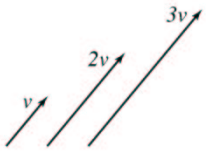
- Vector spaces
- Linear transformations and matrix multiplication
- 2D affine space (Euclidean space)

2D Vector spaces

- Vector
  - an arrow drawn in the plane
  - an offset (position not important)
- Vector operations
  - addition
  - scalar multiplicataion

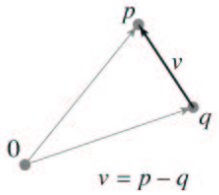






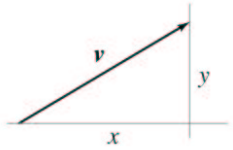
Vectors

- Used to represent points and offsets
- Slightly different datatypes
  - point – point = vector
  - point + vector = point
  - point + point = ?



Representing vectors

- Can use scalars
  - in our case, a pair of real numbers
  - x and y coordinates in some coordinate system
  - addition = add componentwise
  - scaling = multiply both components



The dot product

- Dot product = inner product
- Notation: $\mathbf{u} \cdot \mathbf{v}, \mathbf{u}^T\mathbf{v}, \langle \mathbf{u}, \mathbf{v} \rangle$
- $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
- geometric interpretation: projection
- implementation: $x_u x_v + y_u y_v$

Linear independence and bases

- linear combination: $a\mathbf{u} + b\mathbf{v}$
- linear (in)dependence
  - vectors in same direction: dependent
    - linear combination not useful
  - vectors not in same direction: independent
    - linear combinations can represent all vectors
    - said to be a *basis* for 2D space
- Representing vectors
  - canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$
  - $\begin{bmatrix} x & y \end{bmatrix}^T$ is abbreviation for $x\mathbf{e}_1 + y\mathbf{e}_2$

Defining geometry

- Subsets of the plane
- Generally curves (1D) and areas (2D)
  - related, because the boundaries of areas are curves
- Explicit (parametric) and implicit forms

Implicit representations

- Equation to tell whether we are on the curve
- $\{\mathbf{v} \,|\, f(\mathbf{v}) = 0\}$
- Example: line
  - $\{\mathbf{v} \,|\, \mathbf{v} \cdot \mathbf{u} + k = 0\}$
- Example: circle
  - $\{\mathbf{v} \,|\, (\mathbf{v} - \mathbf{p}) \cdot (\mathbf{v} - \mathbf{p}) + r^2 = 0\}$
- Always define boundary of region
  - (if *f* is continuous)

Explicit representations

- Also called parametric
- Equation to map domain into plane
  - $\{f(t) \,|\, t \in D\}$
- Example: line
  - $\{\mathbf{p} + t\mathbf{u} \,|\, t \in \mathbb{R}\}$
- Example: circle
  - $\{\mathbf{p} + r\begin{bmatrix} \cos t & \sin t \end{bmatrix}^T \,|\, t \in [0, 2\pi)\}$
- Like tracing out the path of a particle over time
- Variable *t* is the "parameter"

Parametric vs. implicit

- Parametric: more direct handle on geometry
  - e.g. easy to generate points on the curve
- Implicit: more global view
  - e.g. easy to tell inside vs. outside
- Many operations equally convenient with either
  - e.g. computing normals or curvature

Transforming geometry

- Move a subset of the plane using a mapping from the plane to itself
  - $S \rightarrow \{T(\mathbf{v}) \,|\, \mathbf{v} \in S\}$
- Parametric representation:
  - $\{f(t) \,|\, t \in D\} \rightarrow \{T(f(t)) \,|\, t \in D\}$
- Implicit representation:
  - $\{\mathbf{v} \,|\, f(\mathbf{v}) = 0\} \rightarrow \{T(\mathbf{v}) \,|\, f(\mathbf{v}) = 0\}$
  - $= \{\mathbf{v} \,|\, T^{-1}(f(\mathbf{v})) = 0\}$

Translation

- Simplest transformation: $T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$
- Inverse: $T^{-1}(\mathbf{v}) = \mathbf{v} - \mathbf{u}$
- Example of transforming circle

Linear transformations

- Any transformation with the property:
  - $T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v})$
- Can be represented using matrix multiplication
  - $T(\mathbf{v}) = M\mathbf{v}$

Matrix-vector multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- Interpretations:
  - sum of scalar products of columns
  - list of dot products with rows

Matrix-matrix multiplication

- Just M-V multiplication for each column

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix}$$

- Key property: $(MN)\mathbf{v} = M(N\mathbf{v})$
- Transpose and column vectors

Geometry of 2D linear trans.

- 2x2 matrices have simple geometric interpretations
  - uniform scale
  - non-uniform scale
  - rotation
  - shear
  - reflection
- Reading off the matrix

2D Geometry and Transformations
CONT'D

CS 417 Lecture 8

Programming hint

```
byte b = 255; // b == -1 == 0b11111111
int i = b; // i == -1 == 0b111…111111111
int i = b & 0xff; // i = 255 = 0b000…011111111
```


Matrix operations

- Matrix transpose: flip rows and columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- Identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  - For all matrices *M*, *IM* = *MI* = *M*

Matrix operations

- Matrix inverse: cancels matrix, leaving identity

$$M^{-1}M = MM^{-1} = I$$

  - inverse undoes whatever *M* does
  - only exists for nonsingular *M* (*M* does not flatten things)

Composing transformations

- Want to move an object, then move it some more
  - $\mathbf{p} \to T(\mathbf{p}) \to S(T(\mathbf{p})) = (S \circ T)(\mathbf{p})$
- We need to represent *S* o *T*
  - and would like to use the same representation as for *S* and *T*
- Translation easy
  - $T(\mathbf{p}) = \mathbf{p} + \mathbf{u}_T; S(\mathbf{p}) = \mathbf{p} + \mathbf{u}_S$
    $(S \circ T)(\mathbf{p}) = \mathbf{p} + (\mathbf{u}_T + \mathbf{u}_S)$
    - commutative!

Composing transformations

- Linear transformations also straightforward
  - $T(\mathbf{p}) = M_T\mathbf{p}; S(\mathbf{p}) = M_S\mathbf{p}$
    $(S \circ T)(\mathbf{p}) = M_S M_T \mathbf{p}$
    - only sometimes commutative
      - e.g. rotations & uniform scales
      - e.g. non-uniform scales w/o rotation

Combining linear with translation

- Need to use both in single framework
- Can represent arbitrary seq. as $T(\mathbf{p}) = M\mathbf{p} + \mathbf{u}$
  - $T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T$
  - $S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S$
  - $(S \circ T)(\mathbf{p}) = M_S(M_T\mathbf{p} + \mathbf{u}_T) + \mathbf{u}_S$
    $= (M_SM_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S)$
  - e. g. $S(T(0)) = S(\mathbf{u}_T)$
- This will work but is a little awkward

Homogeneous coordinates

- A trick for representing the foregoing simply
- Extra component *w* for vectors, extra row/column for matrices
  - for affine, can always keep *w* = 1
- Represent linear transformations with dummy extra row and column

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

Homogeneous coordinates

- Represent translation using the extra column

$$\begin{bmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t \\ y + s \\ 1 \end{bmatrix}$$

Homogeneous coordinates

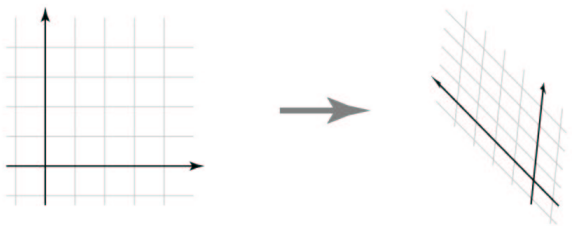
- Composition just works, by 3x3 matrix multiplication

$$\begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} (M_SM_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S) \\ 1 \end{bmatrix}$$

- This is exactly the same as carrying around $M\mathbf{p} + \mathbf{u}$
  - but cleaner
  - and generalizes in useful ways as we'll see later

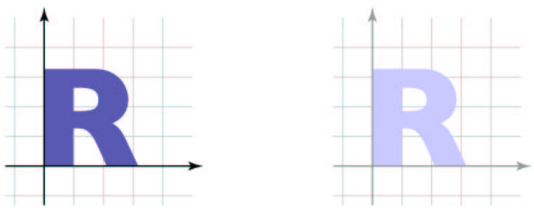
Affine transformations

- The set of transformations we have been looking at is known as the "affine" transformations
  - straight lines preserved; parallel lines preserved
  - ratios of lengths along lines preserved (midpoints preserved)



Affine transformation gallery

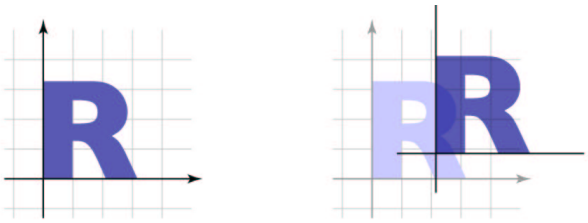
- Translation $\begin{bmatrix} 0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 2.15 \\ 0 & 0 & 0.85 \\ 0 & 0 & 1 \end{bmatrix}$



Affine transformation gallery

- Translation

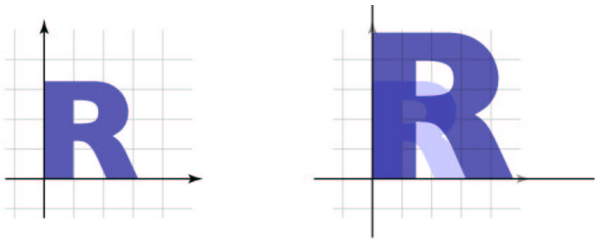
$$\begin{bmatrix} 0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 2.15 \\ 0 & 0 & 0.85 \\ 0 & 0 & 1 \end{bmatrix}$$



Affine transformation gallery

- Uniform scale

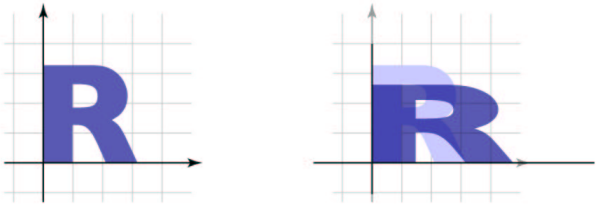
$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Affine transformation gallery

- Nonuniform scale

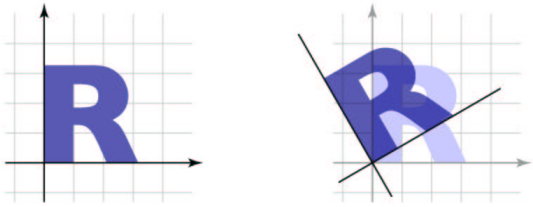
$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Affine transformation gallery

- Rotation

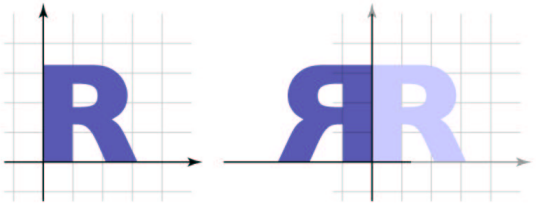
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Affine transformation gallery

- Reflection
  - can consider it a special case of nonuniform scale

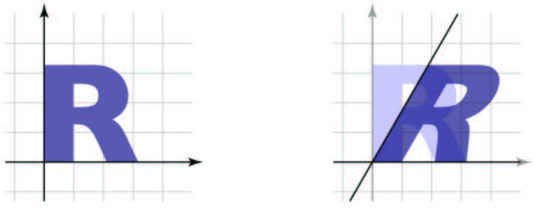
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Affine transformation gallery

- Shear

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

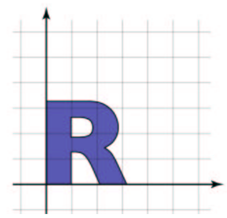


General affine transformations

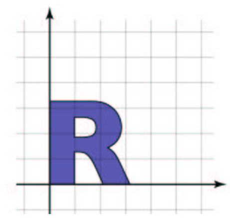
- The previous slides showed “canonical” examples of the types of affine transformations
- Generally, transformations contain elements of multiple types
  - often define them as products of canonical transforms
  - sometimes work with their properties more directly

Composite affine transformations

- In general **not** commutative: order matters!



rotate, then translate



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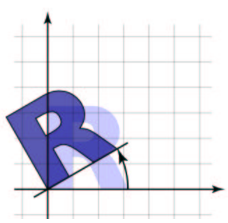

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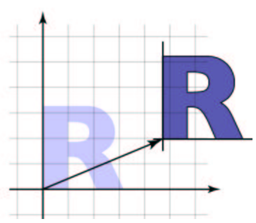

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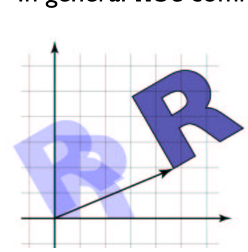

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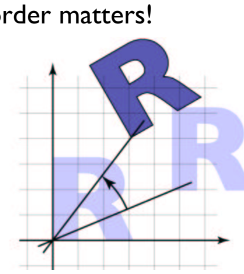

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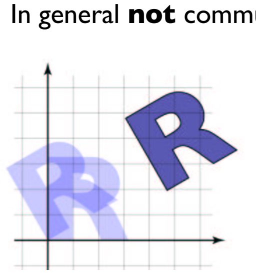

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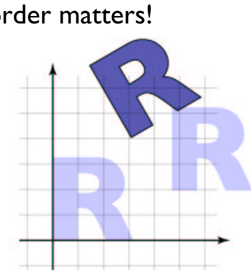
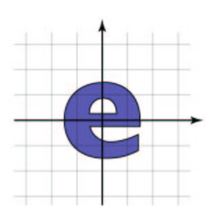


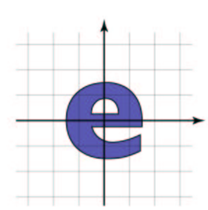
translate, then rotate

Composite affine transformations

- Another example



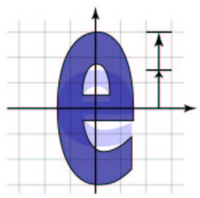
scale, then rotate



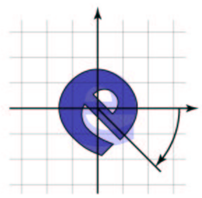
rotate, then scale

Composite affine transformations

- Another example



scale, then rotate



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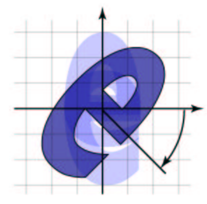

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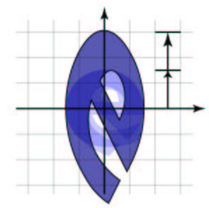

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Composite affine transformations

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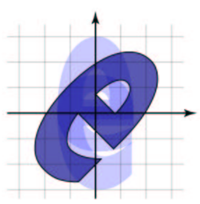

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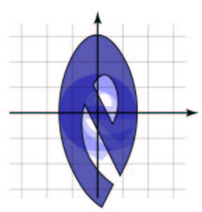


rotate, then scale

Rigid motions

- A transform made up of only translation and rotation is a *rigid motion* or a *rigid body transformation*
- The linear part is an orthonormal matrix

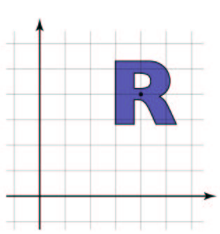
$$R = \begin{bmatrix} Q & \mathbf{u} \\ 0 & 1 \end{bmatrix}$$

- Inverse of orthonormal matrix is transpose
  - so inverse of rigid motion is easy:

$$R^{-1}R = \begin{bmatrix} Q^T & -Q^T\mathbf{u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q & \mathbf{u} \\ 0 & 1 \end{bmatrix}$$

Composing to change axes

- Want to rotate about a particular point
  - could work out formulas directly…
- Know how to rotate about the origin
  - so translate that point to the origin



$$M = T^{-1}RT$$

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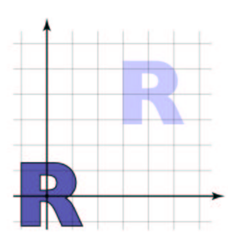

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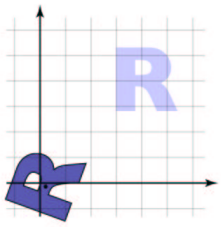

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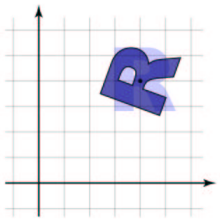

$$M = T^{-1}RT$$

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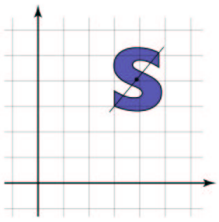
Composing to change axes

- Want to scale along a particular axis and point
- Know how to scale along the y axis at the origin
  - so translate to the origin and rotate to align axes



$$M = T^{-1}R^{-1}SRT$$

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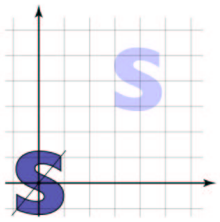

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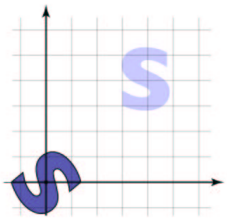
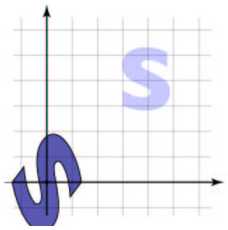


$$M = T^{-1}R^{-1}SRT$$

Composing to change axes

- Want to scale along a particular axis and point
- Know how to scale along the y axis at the origin
  - so translate to the origin and rotate to align axes



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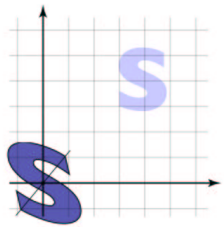

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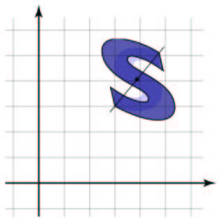

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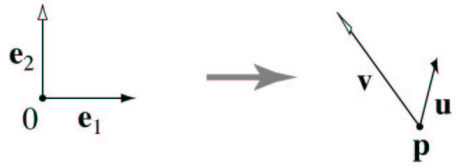
Affine change of coordinates

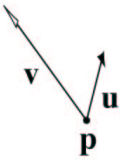
- Six degrees of freedom

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$



Affine change of coordinates

- Coordinate frame: point plus basis
- Interpretation: transformation changes representation of point from one basis to another
- "Frame to canonical" matrix has frame in columns
  - takes points represented in frame
  - represents them in canonical basis
  - e.g. [0 0], [1 0], [0 1]
- Seems backward but bears thinking about



$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

Affine change of coordinates

- When we move an object to the origin to apply a transformation, we are really changing coordinates
  - the transformation is easy to express in object's frame
  - so define it there and transform it

$$T_e = FT_F F^{-1}$$

  - T_e is the transformation expressed wrt. $\{e_1, e_2\}$
  - T_F is the transformation expressed in natural frame
  - F is the frame-to-canonical matrix $[u\ v\ p]$
- This is a *similarity transformation*

Orthonormal frames

- If the frame matrix F is a rigid motion, then the frame is an *orthonormal frame*
  - this just means u and v are perpendicular and unit length
- This is actually the common case, because orthonormal frames make things convenient
  - computationally: easy to invert
  - intellectually: easier to think about

Coordinate frame summary

- Frame = point plus basis
- Frame matrix (frame-to-canonical) is

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{p} \\ 0 & 0 & 1 \end{bmatrix}$$

- Move points to and from frame by multiplying with *F*

$$p_e = F p_F \quad p_F = F^{-1} p_e$$

- Move transformations using similarity transforms

$$T_e = F T_F F^{-1} \quad T_F = F^{-1} T_e F$$

