Prove the Substitution lemma:

Lemma 1 (Substitution). *If* $\{x \mapsto \tau'\} \vdash e_1 : \tau$ *and* $\vdash e_2 : \tau'$ *then* $\vdash e_1[e_2/x] : \tau$.

For the proof to go through, we need to strengthen the lemma; otherwise, we will get stuck on the inductive case where $e_1 = \lambda y.\, e$ with $y \neq x$. To handle this, we introduce more typing assumptions to the lemma. So, instead, we will prove the following stronger lemma:

Lemma 2. *If* $\Gamma[x \mapsto \tau'] \vdash e_1 : \tau$ *and* $\vdash e_2 : \tau'$ *then* $\Gamma \vdash e_1[e_2/x] : \tau$.

Proof. By induction on the height of the proof $\mathcal{P}$ showing $\Gamma[x \mapsto \tau'] \vdash e_1 : \tau$.

Base cases. There are three cases to consider: $e_1 = c$, $e_1 = x$ and $e_1 = y \neq x$.

- If $e_1 = c$, then $\tau = b$. From the definition of substitution, $e_1[e_2/x] = c$. Hence, by the const typing rule, $\Gamma \vdash e_1[e_2/x] : \tau$.
- If $e_1 = x$, then $\tau = \tau'$. From the definition of substitution, $e_1[e_2/x] = e_2$. By assumption, $\vdash e_2 : \tau'$; hence, $\Gamma \vdash e_2 : \tau' = \tau$.
- If $e_1 = y$, then from the definition of substitution, $e_1[e_2/x] = e_1$. If $\Gamma[x \mapsto \tau'] \vdash y : \tau$, then by the var typing rule, $\tau = \Gamma[x \mapsto \tau'](y) = \Gamma(y)$; hence, $\Gamma \vdash e_1[e_2/x] : \tau$.

Induction hypothesis. Assume that the lemma holds for proofs $\mathcal{P}$ of height at most k.

Inductive step. Given the induction hypothesis, we now show that the lemma holds for proofs $\mathcal{P}$ of height $k + 1$.

There are three cases to consider: $e_1 = \lambda x.\, e$, $e_1 = \lambda y.\, e$ with $y \neq x$, and $e_1 = e\ e'$.

- Suppose $e_1 = \lambda x.\, e$. Then $\tau = \tau_1 \to \tau_2$ for some τ_1 and τ_2. From the definition of substitution, $(\lambda x.\, e)[e_2/x] = \lambda x.\, e$. That is, $e_1[e_2/x] = e_1$. By assumption, we have $\Gamma[x \mapsto \tau'] \vdash \lambda x.\, e : \tau_1 \to \tau_2$. By the lam typing rule, this proof must have the form

$$\frac{\dfrac{\mathcal{P}_1}{\Gamma[x \mapsto \tau'][x \mapsto \tau_1] \vdash e : \tau_2}}{\Gamma[x \mapsto \tau'] \vdash \lambda x.\, e : \tau_1 \to \tau_2}.$$

  Noting that $\Gamma[x \mapsto \tau'][x \mapsto \tau_1] = \Gamma[x \mapsto \tau_1]$ and applying the lam typing rule, we get

$$\frac{\dfrac{\mathcal{P}_1}{\Gamma[x \mapsto \tau_1] \vdash e : \tau_2}}{\Gamma \vdash \lambda x.\, e : \tau_1 \to \tau_2}.$$

- Suppose $e_1 = \lambda y.\, e$ with $y \neq x$. Then $\tau = \tau_1 \to \tau_2$ for some τ_1 and τ_2. From the definition of substitution, $(\lambda y.\, e)[e_2/x] = \lambda y.\, (e[e_2/x])$. By assumption, we have $\Gamma[x \mapsto \tau'] \vdash \lambda y.\, e : \tau_1 \to \tau_2$. By the lam typing rule, this proof must have the form

$$\frac{\dfrac{\mathcal{P}_2}{\Gamma[x \mapsto \tau'][y \mapsto \tau_1] \vdash e : \tau_2}}{\Gamma[x \mapsto \tau'] \vdash \lambda y.\, e : \tau_1 \to \tau_2}.$$

Noting that $\Gamma[x \mapsto \tau'][y \mapsto \tau_1] = \Gamma[y \mapsto \tau_1][x \mapsto \tau']$ and letting $\Gamma' = \Gamma[y \mapsto \tau_1]$, we have

$$\frac{\mathcal{P}_2}{\Gamma'[x \mapsto \tau'] \vdash e : \tau_2}.$$

Since $\mathcal{P}_2$ must have height at most k, the induction hypothesis applies, thus giving us $\Gamma' \vdash e[e_2/x] : \tau_2$. Applying the lam typing rule, we get

$$\frac{\Gamma[y \mapsto \tau_1] : e[e_2/x] : \tau_2}{\Gamma : \lambda y.\, (e[e_2/x]) : \tau_1 \to \tau_2}.$$

- Suppose $e_1 = e\ e'$. From the definition of substitution, $(e\ e')[e_2/x] = (e[e_2/x]\ e'[e_2/x])$. By assumption, we have $\Gamma[x \mapsto \tau'] \vdash e\ e' : \tau$. By the app typing rule, this proof must have the form

$$\frac{\dfrac{\mathcal{P}_3}{\Gamma[x \mapsto \tau'] \vdash e : \tau'' \to \tau} \qquad \dfrac{\mathcal{P}_4}{\Gamma[x \mapsto \tau'] \vdash e' : \tau''}}{\Gamma[x \mapsto \tau'] \vdash e\ e' : \tau}.$$

Since $\mathcal{P}_3$ and $\mathcal{P}_4$ must each have height at most k, the induction hypothesis applies, thus giving us $\Gamma \vdash e[e_2/x] : \tau'' \to \tau$ and $\Gamma \vdash e'[e_2/x] : \tau''$. Applying the app rule yields $\Gamma \vdash e[e_2/x]\ e'[e_2/x] : \tau$.

□